

Approximations of Maximal Controlled Safe Sets for Hybrid Systems*

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Abstract

In the determination of the "maximal safe set" for a hybrid system, the core problem consists in the computation of a maximal controlled invariant set contained in a constraint set for a continuous time dynamical system. In the case of a linear system, we propose a procedure that, on the basis of a controlled invariant set for the exponential discretization of the continuous time system, leads to an arbitrarily good approximation of the maximal controlled invariant set for the continuous time system. The approximating set has the interesting property that the constraints can be satisfied by means of a piecewise constant control. An example of application of the proposed procedure to idle control is illustrated.

keywords: controlled invariant sets, safe sets, linear systems

1 Introduction

In the area of hybrid system control, emphasis has been placed on solving problems with safety specifications, which are described by giving a set of good states within which the controlled hybrid system should evolve. The problem of finding the "maximal safe set", i.e. the set of all initial states guaranteeing that the evolution of the system remains in the good set, can be decomposed into a number of sub-problems, each consisting basically in the computation of a maximal controlled invariant set contained in a set of constraints [5].

For discrete-time linear systems, iterative procedures for the computation of maximal controlled invariant sets are well known in the literature (e.g. [12], [14]). However, these procedures may not converge in a finite number of steps. Inner-approximation algorithms were proposed in [6] that yield tight bounds for an error originating from stopping the procedure after a finite number of steps. For continuous-time linear systems, this problem is still open. A result that could in principle be used to determine an approximation of the maximal controlled invariant set for a continuous-time linear system is that if a set S is controlled invariant for the Euler Approximating System, it is also controlled invariant for the continuous-time system [10]. However, if the control law that makes the

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set S invariant is sampled (for example in digital implementation), there is no guarantee that the trajectory of the continuous-time system remains in S between consecutive sampling times. Hence we need to resort to a different approximation scheme.

The key idea is to compute a controlled safe set instead of a controlled invariant set. A controlled safe set is a set such that, using an appropriate piecewise constant control, the evolution of the controlled continuous-time system satisfies the constraints at all times. In this paper, we show how the exponential discretization of the continuous time system can be used for the derivation of a controlled safe set. In Section 2, we summarize some properties of controlled safe sets. In Section 3, we show that, by an appropriate choice of the sampling time, an arbitrarily good approximation of the maximal controlled invariant set can be obtained. Then, we prove that there is no loss of generality in assuming polyhedral state and input constraints, thus simplifying all the numerical calculations, and give some tools for the computation of controlled safe sets. An example of application of the proposed procedure to the idle speed control problem is illustrated in Section 4. Section 5 contains some concluding remarks.

2 Definition and properties of controlled safe sets

Consider a continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

with state and input constraints

$$x(t) \in \Lambda, u(t) \in U, \forall t \geq 0 \quad (2)$$

and a discrete-time linear system:

$$x(k+1) = A_d x(k) + B_d u(k) \quad (3)$$

with state and input constraints

$$x(k) \in \Lambda, u(k) \in U, \forall k \in \mathbb{N} \quad (4)$$

where $\Lambda \subset \mathbb{R}^n$, $U \subset \mathbb{R}^p$.

We denote with $x(t, x_0, u)$ (resp. $x(k, x_0, u)$) the state evolution of the continuous-time system (1) at time t (resp. of the discrete-time system (3) at time k) with initial state x_0 and control law $u|_{[0,t]}$ (resp. $u|_{[0,k]}$). We abuse of notation by writing either $x(t, x_0, u)$ or $x(t, x_0, u(0))$ if $u(t) = u(0)$, $\forall t \geq 0$.

A control law $u(\cdot)$ is said to be feasible with respect to continuous-time system (1) if $u(t) \in U$, $\forall t \geq 0$ (resp. to discrete-time system (3) if $u(k) \in U$, $\forall k \in \mathbb{N}$).

We first give the standard definition of controlled invariant set and the definition of controlled safe set:

Definition 1 A set $\Omega \subset \Lambda$ is controlled invariant for system (1) (resp. (3)) if $\forall x_0 \in \Omega$ there exists a feasible control law $u(\cdot)$ such that the evolution $x(t, x_0, u) \in \Omega$, $\forall t \geq 0$ (resp. $x(k, x_0, u) \in \Omega$, $\forall k \in \mathbb{N}$) and we say that $u(\cdot)$ makes the set Ω invariant for (1) (resp. (3)). A set $\Omega \subset \Lambda$ is a controlled safe set for system (1) (resp. (3)) if $\forall x_0 \in \Omega$ there exists a feasible control law $u(\cdot)$ such that the evolution $x(t, x_0, u) \in \Lambda$, $\forall t \geq 0$ (resp. $x(k, x_0, u) \in \Lambda$, $\forall k \in \mathbb{N}$) and we say that $u(\cdot)$ makes the set Ω safe for (1) (resp. (3)).

Throughout the paper, we assume that

- **Assumption 1 :** Λ and U are C-sets, i.e. convex, compact sets with the origin in their interior
- **Assumption 2 :** (A, B) asymptotically stabilizable

Let \mathcal{I}_c be the maximal controlled invariant subset of Λ for the continuous-time system (1). By Assumptions 1 and 2, \mathcal{I}_c is a C-set. By definition, any controlled safe set $\Omega \subset \Lambda$ is contained in \mathcal{I}_c . Since the union of controlled safe subsets of Λ is a controlled safe set, there exists a unique maximal controlled safe subset of Λ . The maximal controlled safe set in Λ coincides with \mathcal{I}_c .

Consider now a particular discrete-time system of the form (3), the exponential discretization of the continuous-time system (1) with sampling time $T > 0$:

$$x(k+1) = A_d(T)x(k) + B_d(T)u(k) \quad (5)$$

where $A_d(T) = e^{AT}$ and $B_d(T) = \int_0^T e^{A(T-t)}Bdt$. If Assumption 2 holds, then $(A_d(T), B_d(T))$ is asymptotically stabilizable with respect to system (5), $\forall T \in \check{\mathbb{R}}$, where $\check{\mathbb{R}} = (0, \infty)a.e.$ (see [15]).

In the following, we give the tools for the computation of a set that is controlled invariant for (5), controlled safe for (1) and is an arbitrarily good approximation of \mathcal{I}_c . Moreover, this set has the property that the constraints can be satisfied by means of an appropriately chosen piecewise-constant control law. More precisely, the class of controls we focus on is

$$U_T = \{u_T(\cdot) : u_T(t) = u(kT), \forall t \in [kT, (k+1)T), \\ k \in \mathbb{N} \text{ and } u(kT) \in U, \forall k \in \mathbb{N}\}$$

Definition 2 A set $\Omega \subset \Lambda$ is a T -controlled safe set for the continuous-time system (1) if $\forall x_0 \in \Omega$ there exists a feasible piecewise constant control law $u_T(\cdot) \in U_T$ such that the evolution $x(t, x_0, u_T) \in \Omega, \forall t \geq 0$.

Let $S(T)$ be a controlled invariant C-subset of Λ for the exponential discretization system (5). Given $x \in S(T)$, set

$$U(x) = \{u \in U : A_d(T)x + B_d(T)u \in S(T)\} \quad (6)$$

By definition, $\forall x_0 \in S(T)$ there exists a control law $u_T(\cdot) \in U_T$ such that $x(t, x_0, u_T) \in \Lambda, \forall t = kT, k \in \mathbb{N}$. However, since in general $S(T)$ is not controlled invariant for the continuous-time system (1), this is not true for $t \neq kT$. We therefore introduce a precise measure of how much the state evolution of the continuous-time system deviates from the constraint.

Definition 3 Let $S(T)$ be a controlled invariant C-subset of Λ for system (5). The expanding factor $\mu(S(T))$ is the minimum scalar value greater than or equal to one, such that $\forall x_0 \in S(T)$, there exists $u_T(\cdot) \in U_T$ such that the state evolution of system (1) $x(t, x_0, u_T) \in \mu(S(T))\Lambda, \forall t \geq 0$ and $x(t, x_0, u_T) \in S(T), \forall t = kT, k \in \mathbb{N}$.

Given a controlled invariant set for the exponential discretization system (5), a T -controlled safe set can be computed for the continuous-time system (1) using the following:

Proposition 4 Suppose Assumptions 1 and 2 hold and $T \in \check{\mathbb{R}}$. If $S(T)$ is a controlled invariant C-subset of Λ for system (5), then the set $\frac{S(T)}{\mu(S(T))}$ is a T -controlled safe set for the continuous-time system (1). Moreover, if $u'(\cdot)$ is the control law which makes $\frac{S(T)}{\mu(S(T))}$ invariant for the discrete-time system (5), the piecewise constant control $u(t) = u'(kT), \forall t \in [kT, (k+1)T), k \in \mathbb{N}$ makes $\frac{S(T)}{\mu(S(T))}$ safe for the continuous-time system (1).

3 Inner approximations of the maximal controlled invariant set

By Proposition 4, under Assumptions 1 and 2 and $T \in \check{\mathbb{R}}$, if $S(T)$ is a controlled invariant C -subset of Λ for system (5), then $\frac{S(T)}{\mu(S(T))}$ is T -controlled safe set for (1) and is therefore contained in \mathcal{I}_c . This suggests to approximate \mathcal{I}_c with $\frac{\mathcal{I}(T)}{\mu(\mathcal{I}(T))}$ where $\mathcal{I}(T)$ is the maximal controlled invariant subset of Λ for the discrete-time system (5) under constraints (4).

The next result shows that, by an appropriate choice of the sampling time T , $\frac{\mathcal{I}(T)}{\mu(\mathcal{I}(T))}$ can be an arbitrarily good approximation of \mathcal{I}_c . The following definition will be used.

Definition 5 *Let \mathcal{A} and \mathcal{B} be two C -sets. Given $\varepsilon > 0$, \mathcal{B} is an ε -approximation of \mathcal{A} if:*

$$\mathcal{B} \subset \mathcal{A} \subset (1 + \varepsilon)\mathcal{B}$$

Theorem 6 [11] *If Assumptions 1 and 2 hold, then $\forall \varepsilon > 0 \exists \mathbf{T} > 0$ such that $\frac{\mathcal{I}(T)}{\mu(\mathcal{I}(T))}$ is an ε -approximation of $\mathcal{I}_c, \forall T \in (0, \mathbf{T}] \cap \check{\mathbb{R}}$.*

We now show that there is no loss of generality in assuming that the C -sets Λ and U are polyhedral. Given the C -sets Λ and U , consider the sets Λ_p and U_p , respectively polyhedral ε -approximations of Λ and U , i.e. for any given $\varepsilon > 0$,

$$\begin{aligned} \Lambda_p &\subset \Lambda \subset (1 + \varepsilon)\Lambda_p \\ U_p &\subset U \subset (1 + \varepsilon)U_p \end{aligned} \tag{7}$$

Consider the discrete-time system (5) with state and input constraints Λ_p and U_p , i.e.

$$x(k) \in \Lambda_p, u(k) \in U_p, \forall k \in \mathbb{N} \tag{8}$$

Let $\mathcal{I}_p(T)$ be the maximal controlled invariant set for discrete-time system (5) under constraints (8).

By definition, if (7) holds, we have

$$\mathcal{I}_p(T) \subset \mathcal{I}(T) \subset (1 + \varepsilon)\mathcal{I}_p(T)$$

In [6] it is shown that, under Assumptions 1 and 2 and $T \in \check{\mathbb{R}}$, $\mathcal{I}(T)$ and $\mathcal{I}_p(T)$ are C -sets. However, $\mathcal{I}_p(T)$ is not necessarily polyhedral, but for any given $\varepsilon > 0$ there exists a polyhedral controlled invariant set $\mathcal{P}(T)$, for system (5) under constraints (8), which satisfies

$$\mathcal{P}(T) \subset \mathcal{I}_p(T) \subset (1 + \varepsilon)\mathcal{P}(T) \tag{9}$$

As a consequence of relation (7), the sets $\mathcal{I}_p(T)$ and $\mathcal{P}(T)$ are controlled invariant with respect to system (5) under constraints (4). Therefore the sets $\frac{\mathcal{I}_p(T)}{\mu(\mathcal{I}_p(T))}$ and $\frac{\mathcal{P}(T)}{\mu(\mathcal{P}(T))}$ are T -controlled safe sets for continuous-time system (1) under constraints (2). Moreover, the following holds:

Theorem 7 [4] *Suppose Assumptions 1 and 2 hold and $T \in \check{\mathbb{R}}$. $\forall \delta > 0, \exists \varepsilon > 0$ such that, if $\mathcal{P}(T)$ is a polyhedral controlled invariant set for the discrete-time system (5) satisfying*

$$\mathcal{P}(T) \subset \mathcal{I}(T) \subset (1 + \varepsilon)\mathcal{P}(T) \tag{10}$$

then

$$\mu(\mathcal{P}(T)) - \delta \leq \mu(\mathcal{I}(T)) \leq \mu(\mathcal{P}(T)) + \delta \quad (11)$$

and

$$\frac{\mathcal{P}(T)}{\mu(\mathcal{P}(T)) + \delta} \subset \frac{\mathcal{I}(T)}{\mu(\mathcal{I}(T))} \subset (1 + \varepsilon) \frac{\mathcal{P}(T)}{\mu(\mathcal{P}(T)) - \delta} \quad (12)$$

As a consequence of Theorem 7, we can assume hereafter that

- **Assumption 3 :** $\Lambda = \Lambda_p$ and $U = U_p$

In the following subsection a procedure is presented in order to compute a polyhedral ε -approximation of $\mathcal{I}(T)$ which, under Assumption 3, coincides with $\mathcal{I}_p(T)$.

In subsection 3.2, some tools are given for the computation of the expanding factor.

3.1 Computation of a polytopic approximating set of $\mathcal{I}(T)$

We illustrate here some results, established in [6], which give the means for the computation of an ε -approximation of $\mathcal{I}(T)$.

Proposition 8 [6] *Suppose Assumptions 1 and 2 hold and $T \in \check{\mathbb{R}}$. Define the sequence of sets $\{\Lambda_i, i = 0, 1, 2, \dots\}$*

$$\begin{aligned} \Lambda_i &= \Lambda \quad i = 0 \\ \Lambda_i &= \{x : \exists u \in U : A_d(T)x + B_d(T)u \subset \Lambda_{i-1}\} \cap \Lambda \quad i > 0 \end{aligned} \quad (13)$$

This sequence converges, in general asymptotically, to the set $\mathcal{I}(T)$. $\forall \varepsilon > 0 \exists \mathbf{i}$ such that $\mathcal{I}(T)$ is an ε -approximation of $\Lambda_i, \forall i \geq \mathbf{i}$.

The recursion (13) is not guaranteed to terminate in a finite number of steps. Each of the sets Λ_i contains the set $\mathcal{I}(T)$ and hence can be viewed as an *outer approximation* of $\mathcal{I}(T)$. Obviously this approximation becomes better and better as i increases. However, the sets Λ_i are not controlled invariant, and hence they are not useful from a control synthesis point of view.

We then resorted in [6] to give algorithms for the computation of *inner controlled invariant approximations* of the set $\mathcal{I}(T)$, as illustrated below.

Suppose Assumptions 1,2 and 3 hold and $T \in \check{\mathbb{R}}$. Let S be a controlled invariant C-subset of Λ for system (5). Define the sequence $\{\Omega^i(S), i = 0, 1, 2, \dots\}$, where

$$\begin{aligned} \Omega_0(S) &= S \\ \Omega_i(S) &= \left\{ \begin{array}{l} x : x_0 = x, x(i, x_0, u) \in S, x(k, x_0, u) \in \Lambda, \forall k \quad 0 \leq k \leq i \\ \text{for some } u = \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(i-1) \end{pmatrix}, \text{ where } u(j) \in U, 0 \leq j \leq i-1 \end{array} \right\}. \end{aligned}$$

The sets $\Omega_i(S)$ are controlled invariant $\forall i \geq 0$ for system (5). Moreover, they can be computed by means of the following procedure:

$$\begin{aligned}\Omega_i(S) &= S \quad i = 0 \\ \Omega_i(S) &= \{x : \exists u \in U : A_d(T)x + B_d(T)u \subset \Omega_{i-1}(S)\} \cap \Lambda \quad i > 0\end{aligned}\tag{14}$$

The following procedure computes an ε -approximation of $\mathcal{I}(T)$.

Procedure 9 [6] *Suppose Assumptions 1,2 and 3 hold and $T \in \check{\mathbb{R}}$. Let $\varepsilon > 0$ be given.*

1. *Compute a controlled invariant C-set $S \subset \Lambda$ for system (5) (as shown e.g. in [13] and [7]).*
2. *$i = 0$;
repeat $i = i + 1$; $\mathcal{P}(T) = \Omega_i(S)$
until $\mathcal{P}(T)$ is an ε -approximation of Λ_i , and hence an ε -approximation of $\mathcal{I}(T)$.*

Theorem 10 [6] *Suppose Assumptions 1,2 and 3 hold and $T \in \check{\mathbb{R}}$. For any given $\varepsilon > 0$, Procedure 9 terminates in a finite number of steps. The computed set $\mathcal{P}(T)$ is controlled invariant for system (5) and is an ε -approximation of $\mathcal{I}(T)$.*

3.2 Computation of the expanding factor

We summarize here some results from [4] which establish some properties of the expanding factor and give the means for its computation.

Given a C-set Ω and $x \in \mathbb{R}^n$, consider the so-called Minkowski functional [9]:

$$\Psi_\Omega(x) = \inf \{ \mu \in \mathbb{R}, \mu \geq 0 : x \in \mu\Omega \}.$$

>From this definition, Ω can be thought as a unit ball and $x \in \Omega$ if and only if $\Psi_\Omega(x) \leq 1$. The Minkowski functional satisfies the following properties:

$$\begin{aligned}\forall x, y \in \mathbb{R}^n, \\ \Psi_\Omega(x) &\geq 0; \\ \Psi_\Omega(\lambda x) &= \lambda \Psi_\Omega(x), \quad \forall \lambda \geq 0; \\ \Psi_\Omega(x + y) &\leq \Psi_\Omega(x) + \Psi_\Omega(y).\end{aligned}\tag{15}$$

The following Theorem holds.

Theorem 11 [4] *Suppose Assumptions 1 and 2 hold and $T \in \check{\mathbb{R}}$. Let $S(T)$ be a controlled invariant C-subset of Λ for system (5), then*

$$\mu(S(T)) = \max \left\{ 1, \max_{x_0 \in S(T)} \min_{u_T(0) \in U(x_0)} \max_{t \in [0, T]} \Psi_\Lambda(x(t, x_0, u_T(0))) \right\}$$

where $u_T(\cdot) \in U_T$, $u_T(t) = u_T(0)$, $\forall t \geq 0$ and

$$U(x) = \{u \in U : A_d(T)x + B_d(T)u \in S(T)\}$$

The expanding factor can be rendered arbitrarily close to 1 by appropriately choosing the sampling time T , as stated below.

Theorem 12 [4] Suppose Assumptions 1 and 2 hold and $T \in \check{\mathbb{R}}$. $\forall \varepsilon > 0, \exists \widehat{T} > 0$ such that, if $S(T)$ is a controlled invariant C -subset of Λ for (5), then

$$\mu(S(T)) \leq 1 + \varepsilon, \forall T \leq \widehat{T}.$$

The following theorem specializes the result of Theorem 11 to the polytopic controlled invariant set $\mathcal{P}(T)$. In particular, last part of the proof of Theorem 13 illustrates a step-by-step procedure for the calculation of the expanding factor of $\mathcal{P}(T)$.

Theorem 13 [4] Suppose Assumptions 1,2 and 3 hold and $T \in \check{\mathbb{R}}$. Write

$$\mathcal{P}(T) = \text{Conv} \{v_1, v_2, \dots, v_q\}$$

where v_1, v_2, \dots, v_q are the vertexes of $\mathcal{P}(T)$. Let the convex sets $U(v_i)$ be represented by

$$U(v_i) = \text{Conv} \{u_i^1, u_i^2, \dots, u_i^{N_i}\}, i = 1, 2, \dots, q.$$

Then

$$\mu(\mathcal{P}(T)) = \max \left\{ 1, \max_{i=1,2,\dots,q} \min_{k=1,2,\dots,N_i} \max_{t \in [0,T]} \Psi_\Lambda(x(t, v_i, u_i^k)) \right\}$$

where $u_i^k \in U_T$ and $u_i^k(t) = u_i^k(0), \forall t \geq 0$.

Proof. First we show that

$$\mu(\mathcal{P}(T)) = \max \left\{ 1, \max_{i=1,2,\dots,q} \min_{u_T(0) \in U(v_i)} \max_{t \in [0,T]} \Psi_\Lambda(x(t, v_i, u_T(0))) \right\} \quad (16)$$

where $u_T \in U_T$ and $u_T(t) = u_T(0), \forall t \geq 0$.

For each $x_0 \in \mathcal{P}(T)$, consider $x_0 = \sum_{i=1}^q \lambda_i v_i$, with $\lambda_i \geq 0, \forall i = 1, 2, \dots, q$ and $\sum_{i=1}^q \lambda_i = 1$. In [4] it is shown that

$$\sum_{i=1}^q \lambda_i U(v_i) \subset U(x_0) \quad (17)$$

Then, the following chain of inequalities can be obtained using relation (17), the Minkowski functional properties (15), the operator max and the definition of the coefficients λ_i :

$$\begin{aligned} & \min_{u_T(0) \in U(x_0)} \left(\max_{t \in [0,T]} \Psi_\Lambda(x(t, x_0, u_T(0))) \right) \leq \quad (18) \\ & \leq \min_{u_T(0) \in \sum_{i=1}^q \lambda_i U(v_i)} \left(\max_{t \in [0,T]} \Psi_\Lambda(x(t, x_0, u_T(0))) \right) = \\ & = \min_{u_T(0) = \sum_{i=1}^q \lambda_i u_i \in \sum_{i=1}^q \lambda_i U(v_i)} \left(\max_{t \in [0,T]} \Psi_\Lambda \left(\sum_{i=1}^q \lambda_i x(t, v_i, u_i) \right) \right) \leq \\ & \leq \min_{u_T(0) = \sum_{i=1}^q \lambda_i u_i \in \sum_{i=1}^q \lambda_i U(v_i)} \left(\sum_{i=1}^q \lambda_i \max_{t \in [0,T]} (\Psi_\Lambda(x(t, v_i, u_i))) \right) = \\ & = \sum_{i=1}^q \lambda_i \left(\min_{u_i \in U(v_i)} \left(\max_{t \in [0,T]} (\Psi_\Lambda(x(t, v_i, u_i))) \right) \right) \leq \\ & \leq \max_{i=1,2,\dots,q} \left(\min_{u_i \in U(v_i)} \left(\max_{t \in [0,T]} (\Psi_\Lambda(x(t, v_i, u_i))) \right) \right) \sum_{i=1}^q \lambda_i = \\ & = \max_{i=1,2,\dots,q} \min_{u_T(0) \in U(v_i)} \max_{t \in [0,T]} \Psi_\Lambda(x(t, v_i, u_T(0))) \end{aligned}$$

where $u_i \in U(v_i) \forall i = 1, 2, \dots, q$. Therefore

$$\begin{aligned} & \max_{x_0 \in \mathcal{P}(T)} \min_{u_T(0) \in U(x_0)} \left(\max_{t \in [0, T]} \Psi_\Lambda(x(t, x_0, u_T(0))) \right) \leq \\ & \max_{i=1, 2, \dots, q} \min_{u_T(0) \in U(v_i)} \max_{t \in [0, T]} \Psi_\Lambda(x(t, v_i, u_T(0))) \end{aligned} \quad (19)$$

On the other hand, by definition,

$$\begin{aligned} & \max_{x_0 \in \mathcal{P}(T)} \left(\min_{u \in U(x_0)} \left(\max_{t \in [0, T]} \Psi_\Lambda(x(t, x_0, u)) \right) \right) \geq \\ & \geq \max_{x_0 \in \{v_1, v_2, \dots, v_q\}} \min_{u_T(0) \in U(x_0)} \max_{t \in [0, T]} \Psi_\Lambda(x(t, x_0, u_T(0))) \end{aligned} \quad (20)$$

Combining (20) with (19), (16) holds. We now show that the minimum in relation (16) can be computed with respect to the vertexes of the polytopes $U(v_i)$.

Given some vector v and some matrix M , the symbols v_i and M_i denote, respectively, the i -th component of the vector v and the i -th row of the matrix M . Let the external representation of the polytope Λ be as follows:

$$\Lambda = \{x \in \mathbb{R}^n : Lx \leq l\}$$

where $L \in \mathbb{R}^{s \times n}$, $l \in \mathbb{R}^s$, and $l_i > 0, \forall i = 1, 2, \dots, s$, since Λ is a polyhedral C-set.

For a fixed v_i and $u \in U(v_i)$, set

$$\eta(v_i, u) = \max_{t \in [0, T]} \Psi_\Lambda(x(t, v_i, u))$$

It is easy to show that

$$\eta(v_i, u) = \max_{k=1, \dots, s} \theta_k(v_i, u)$$

where $\theta_k(v_i, u) = \max_{t \in [0, T]} \Phi_k(t, v_i, u)$, with $\Phi_k(t, v_i, u) = \frac{L_k}{l_k} (e^{At} v_i + e^{At} F(t) B u)$ where $F(t) = \int_0^t e^{-A\alpha} d\alpha$.

This optimization problem is solvable by numerical methods, since it is the search of the maximum of a scalar function on a scalar compact domain. Let $t_i^* \in [0, T]$ be the time at which $\Phi_k(t, v_i, u)$ attains the maximum and let $\theta_j(v_i, u)$ be the maximal value among $\theta_k(v_i, u)$, $k = 1, \dots, s$.

Then,

$$\eta(v_i, u) = \frac{L_j}{l_j} e^{At_j^*} v_i + \frac{L_j}{l_j} e^{At_j^*} F(t_j^*) B u$$

We have now to find the minimum with respect to $u \in U(v_i)$, i.e.:

$$\min_{u \in U(v_i)} \eta(v_i, u) = \frac{L_j}{l_j} e^{At_j^*} v_i + \min_{u \in U(v_i)} \frac{L_j}{l_j} e^{At_j^*} F(t_j^*) B u$$

Since $U(v_i)$ is a polytope, the above optimization is a constrained optimization of a linear function on a polytopic domain. From Weierstrass Theorem this minimum exists and from Linear Programming Theory it is on one of the vertexes of $U(v_i)$. Hence we have

$$\begin{aligned} \mu(\mathcal{P}(T)) &= \max_{i=1, 2, \dots, q} \min_{u_T(0) \in U(x_0)} \max_{t \in [0, T]} \Psi_\Lambda(x(t, v_i, u_T(0))) = \\ &= \max_{i=1, 2, \dots, q} \min_{k=1, 2, \dots, N_i} \max_{t \in [0, T]} \Psi_\Lambda(x(t, v_i, u_i^k)) \end{aligned}$$

where $u_i^k \in U(v_i) = \text{Conv} \{u_i^1, u_i^2, \dots, u_i^{N_i}\}$. ■

4 An example of application

The problem of maintaining the crankshaft speed within a given range was formalized as a hybrid system control problem with safety specifications in [2] (see also [3] in this Volume, where the synthesis of an idle control strategy is based on the assume-guarantee paradigm). We consider here a simplified model of the powertrain dynamics with the objective of illustrating the procedures presented in Section 3. Consider the system:

$$\begin{cases} \dot{p} = a_m n p + b_m \alpha \\ \dot{n} = a_n n + b_n T \end{cases} \quad (21)$$

where n is the engine speed expressed in RPM (Revolutions Per Minute), p is the manifold pressure expressed in mbar, a_m , b_m , a_n , b_n , are constants, $T = k_1 \eta(AV) p$ is the torque produced by the engine, given the efficiency function $\eta(AV)$, the spark advance angle AV and the constant k_1 .

We use the following numerical values:

$$\begin{aligned} a_n &= -1.5308, & b_n &= 95.4930 \\ a_m &= 0.0262, & b_m &= 1.8212e3 \end{aligned}$$

The two control inputs are α -the throttle opening angle- and AV -the spark advance angle.

The idle-speed control problem consists in finding under which conditions it is possible to maintain the engine speed into the desired range 800 ± 30 RPM, with the following constraints on the control inputs:

$$\begin{aligned} 0^\circ &\leq \alpha \leq 20^\circ \\ 0^\circ &\leq AV \leq 20^\circ \end{aligned}$$

In order to solve the problem, we need to linearize (21) and then compute the exponential discretization. The linearization is made around the operating point $n = 800$ RPM, $p = 300$ mBar, $\alpha = 3.45$ degrees, $AV = 5.75$ degrees. We then consider the exponential discretization of the resulting continuous time linear system with $T = 0.05$.

Using the techniques presented in [6], the maximal controlled invariant set $\mathcal{I}(T)$ for the discrete-time system (5) is found; the set $\mathcal{I}(T)$ is a polytope and its external representation is given by

$$\begin{bmatrix} 1.0000 & 0 \\ -1.0000 & 0 \\ 0 & -1.0000 \\ 0.9920 & 0.0193 \\ 0.9833 & 0.0365 \\ 0.9741 & 0.0519 \\ 0.9644 & 0.0655 \\ 0.9543 & 0.0776 \\ 0.9438 & 0.0882 \\ 0.9329 & 0.0976 \\ 0.9218 & 0.1059 \\ 0.9105 & 0.1131 \\ 0.8991 & 0.1194 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 30 \\ 30 \\ 300 \\ 31.1430 \\ 32.8510 \\ 35.0615 \\ 37.7175 \\ 40.7679 \\ 44.1660 \\ 47.8698 \\ 51.8411 \\ 56.0454 \\ 60.4518 \end{bmatrix}$$

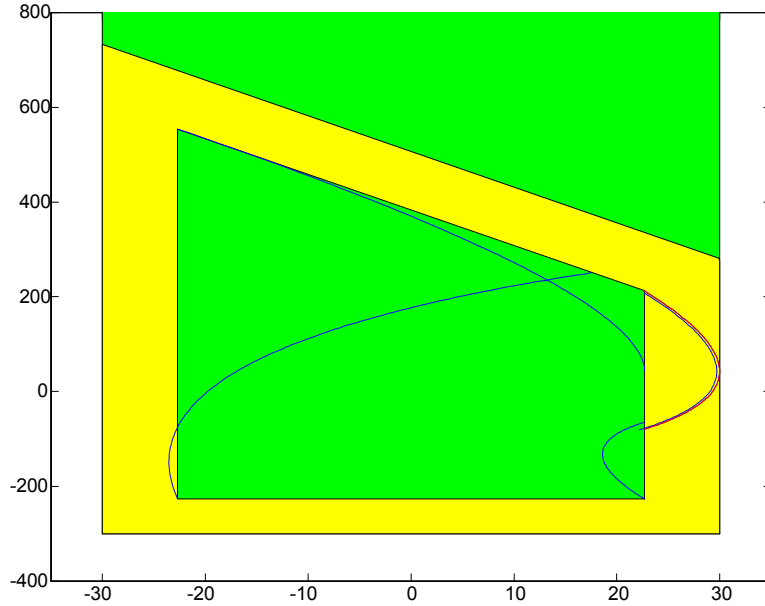


Figure 1:

where

$$\begin{aligned} x_1 &= n - 800 \\ x_2 &= p - 300 \end{aligned}$$

Since the state and input constraints are polyhedral and $\mathcal{I}(T)$ is a polytope, we can directly apply the techniques presented in Theorem 13, obtaining:

$$\mu(\mathcal{I}(T)) = 1.3220$$

The controlled safe set is:

$$\mathcal{I}(T)/\mu(\mathcal{I}(T)) = \mathcal{I}(T)/1.3220$$

In Figure 1, the outer polytope represents the state constraining set Λ ; the inner polytope is $\mathcal{I}(T)/\mu(\mathcal{I}(T))$ and the polytope in between the other two is $\mathcal{I}(T)$. Some trajectories starting from the vertexes of $\mathcal{I}(T)/\mu(\mathcal{I}(T))$ and using some piecewise control law $u \in U(v_i), i = 1, 2, \dots, q$ are also shown. It can be seen that such trajectories do not violate the constraints on the state values. In Figure 2, the values of $\mu(\mathcal{I}(T))$ are plotted for $T \in [0.05, 0.45]$. For the problem at hand, we obtain the smallest expanding factor $\mu(\mathcal{I}(T)) = 1.0154$ with a sampling period of $T = 0.003$; smaller sampling times give rise to numerical problems.

5 Conclusions

In this paper, we have proposed approximating the maximal controlled invariant set contained in a set of constraints for a continuous-time system with a T -controlled safe set. We have shown that

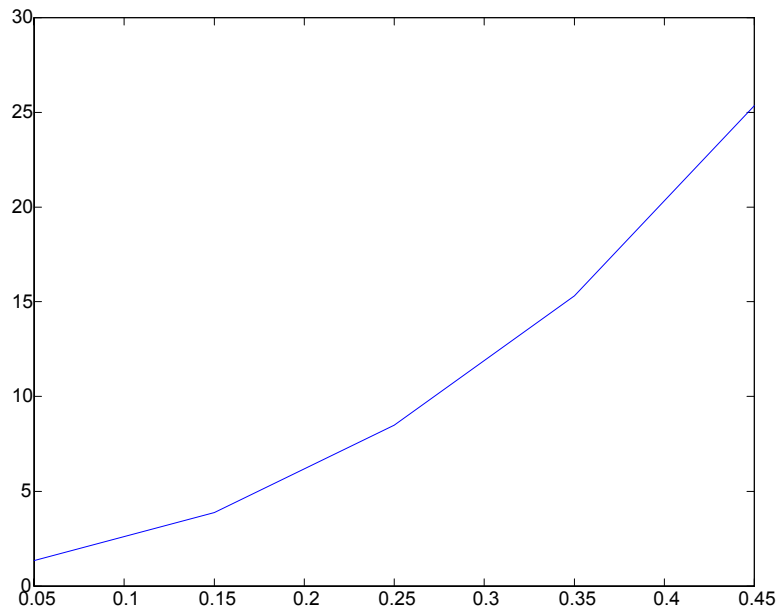


Figure 2:

a T -controlled safe set can be obtained by scaling a controlled invariant set for the exponential discretization system by the expanding factor. Moreover, we have also precisely determined how far from the maximal controlled invariant set our approximating set is. An application to the idle speed control problem has been illustrated.

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