A symbolic model approach to the digital control of nonlinear time–delay systems

Giordano Pola, Pierdomenico Pepe, Maria D. Di Benedetto and Paulo Tabuada

Abstract— **In this paper we propose an approach to control design of nonlinear time–delay systems, which is based on the construction of symbolic models, where each symbolic state and each symbolic label correspond to an aggregate of continuous states and to an aggregate of input signals in the original system. The use of symbolic models offers a systematic methodology for control design in which constraints coming from software and hardware, interacting with the physical world, can be integrated. The main contribution of this paper is in showing that incrementally input–to–state stable time–delay systems do admit symbolic models that are approximately bisimilar to the original system, with a precision that can be rendered as small as desired. An algorithm is also presented which computes the proposed symbolic models. When the state and input spaces of time–delay systems are bounded the proposed algorithm is shown to terminate in a finite number of steps.**

keywords: Time–delay systems, symbolic models, approximate bisimulation, incremental stability.

I. INTRODUCTION

Time–delay systems are an important class of dynamical systems which have been the subject of intensive study during the last years since they model important classes of processes arising in biology, chemical, electrical, mechanical engineering, economics and etc. (see e.g. [1], [2]). Time–delay systems are also relevant in the design of embedded systems which are often characterized by delays in the microprocessor computations and in the exchange of information through communication networks.

Current literature on nonlinear time–delay systems mainly focuses on stabilization, regulation and linearization problems, and important results were achieved (see e.g. [2]). However, the constant evolution of technology demands to make similar progress with respect to control design with more complex specifications, like safety properties, liveness properties, among many others (see e.g. [3]).

In this paper we propose an approach to the control design of nonlinear time-delay systems, based on symbolic models. Symbolic models are abstract models where each symbolic state and each symbolic label represent an aggregation of continuous states and an aggregation of input signals in the original model. Since these symbolic models are of the same nature of the models used in computer science to describe software and hardware, they provide a unified language to study problems of control in which software and hardware interact with the physical world. Moreover, the use of symbolic models allows one to leverage the rich literature developed in the computer science community, as for example supervisory control [4] and algorithmic game theory [5], for control design of purely continuous processes. The crucial step in this approach is the construction of symbolic models that are approximately equivalent to time–delay systems. The notion of approximate equivalence that we consider is *approximate bisimulation*, recently introduced in [6] and [7]. Approximate bisimulation reformulates the classical notion of bisimulation as introduced by Milner and Park [8], [9] in an approximating settings. While (exact) bisimulation as in [8], [9] requires that observations of the states are identical, the notion of approximate bisimulation relaxes this condition, by allowing observations to be close and within a desired precision. This more flexible notion of bisimulation allows one to identify larger classes of systems admitting symbolic models, as for example incrementally stable nonlinear control systems, recently shown in the work of [10], [11].

The main contribution of this paper is in showing that incrementally stable time–delay systems do admit symbolic models that are approximately bisimilar to the original system, with a precision that can be rendered as small, as desired. The proposed symbolic models are shown to be effectively constructed and in fact an algorithm is presented which outputs symbolic models for incrementally stable time–delay systems. When the state and input spaces of the time–delay system are bounded, which is the case in many realistic situations, the proposed algorithm is proved to converge in a finite number of steps. The proofs of the results presented in this paper are omitted for lack of space. A full version of the paper can be found in [12]. In this paper we will use a notation which is standard within both the control and computer science community. However for the sake of completeness, a detailed list of the employed notation is included in the Appendix.

II. TIME–DELAY SYSTEMS

In this paper we consider the following nonlinear time–delay system:

$$\begin{cases} \dot{x}(t) = f(x_t, u(t-r)), & t \in \mathbb{R}^+, a.e. \\ x(t) = \xi_0(t), & t \in [-\Delta, 0], \end{cases} \tag{1}$$

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where $\Delta \in \mathbb{R}_0^+$ is the maximum involved state delay, $r \in \mathbb{R}_0^+$ is the input delay, $x(t) \in X \subseteq \mathbb{R}^n$, $x_t \in \mathcal{X} \subseteq C^0([-\Delta, 0]; X)$, $u(t) \in U \subseteq \mathbb{R}^m$ is the control input at time $t \in [-r, +\infty[$, $\xi_0 \in \mathcal{X}$ is the initial condition, f is a functional from $\mathcal{X} \times U$ to X. We denote by $\mathcal{U}$ the class of control input signals and we suppose that $\mathcal{U}$ is a subset of the set of all measurable and locally essentially bounded functions of time from $[-r, +\infty[$ to U. Moreover we suppose that f is Lipschitz on bounded sets, i.e. for every bounded set $K \subset \mathcal{X} \times U$, there exists a constant $\kappa > 0$ such that

$$\|f(x_1, u_1) - f(x_2, u_2)\| \leq \kappa(\|x_1 - x_2\|_\infty + \|u_1 - u_2\|),$$

for all $(x_1, u_1), (x_2, u_2) \in K$. Without loss of generality we assume $f(0, 0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t, 0)$.

As it is well known, the dependence of the functional f on x_t allows one to consider a very broad class of systems. For instance, the system:

$$\begin{cases} \dot{x}(t) = & \bar{f}(x(t), x(t - \Delta_1), \dots, x(t - \Delta_P), \\ & \int_{-\Delta}^{0} A(\theta, x(t + \theta))d\theta, u(t - r)), \qquad t \in \mathbb{R}^+, a.e. \\ x(t) = & \xi_0(t), \qquad t \in [-\Delta, 0], \end{cases} \tag{2}$$

where $P \in \mathbb{N}$, $\bar{f} : X^{P+2} \times U \to X$, $A : [-\Delta, 0] \times X \to X$ are suitable functions (not functionals), can be cast into the framework of the system in (1). For seeing this just recall that for any real $s \geq 0$, $x(t - s) = x_t(-s)$. The time-delays $\Delta_1, \dots, \Delta_P$ are called discrete time-delays. These discrete time-delays are arbitrary and can be non-commensurate, i.e. a positive real s such that $\Delta_i = j_i s$, with $j_i \in \mathbb{N}$, $i = 1, \dots, P$, does not exist. The term $\int_{-\Delta}^{0} A(\theta, x(t + \theta))d\theta$ is called distributed delay term. Therefore, multiple discrete, arbitrary (also non-commensurate) time-delays as well as distributed delay terms can appear in the system of (1).

Assumptions on f ensure existence and uniqueness of the solutions of the differential equation in (1). In the following $x(t, \xi_0, u)$ and $x_t(\xi_0, u)$ will denote the solutions in X and respectively in $\mathcal{X}$, of the time–delay system with initial condition ξ_0 and input $u \in \mathcal{U}$, at time t. A time–delay system is said to be *forward complete* if every solution is defined on $[0, +\infty[$. In what follows, the time–delay system in (1) is represented by:

$$\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}, f),$$

where each entity is defined as before. The results presented in this paper will assume a stability assumption which we introduce hereafter.

Definition 1: A time–delay system $\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}, f)$ is *incrementally input–to–state stable* (δ–ISS) if it is forward complete and there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that for any time $t \in \mathbb{R}_0^+$, any initial conditions $\xi_1, \xi_2 \in \mathcal{X}$ and any inputs $u_1, u_2 \in \mathcal{U}$ the following inequality holds:

$$\begin{aligned} \|x_t(\xi_1, u_1) - x_t(\xi_2, u_2)\|_\infty \leq \; & \beta(\|\xi_1 - \xi_2\|_\infty, t) \\ & + \gamma(\|(u_1 - u_2)|_{[-r, t-r)}\|_\infty). \end{aligned}$$

The above definition can be thought of as an incremental version of the notion of input–to–state stability (ISS). Since $f(0, 0) = 0$ it is readily seen that δ–ISS implies ISS, by comparing a solution of Σ with initial condition ξ_1 and control input u_1 with the trivial solution. On the other hand, the converse is not true in general, see e.g. some counterexamples in [13]. In general, inequality in (3) is difficult to check directly. A sufficient condition which is based on Liapunov–Krasovskii [14], [15], [16] functionals, can be found in [12].

III. SYMBOLIC MODELS AND APPROXIMATE EQUIVALENCE

In this paper we use transition systems as abstract mathematical models of time–delay systems.

Definition 2: A transition system is a sixtuple:

$$T = (Q, q_0, L, \longrightarrow, O, H),$$

consisting of:

- A set of states Q;
- An initial state $q_0 \in Q$;
- A set of labels L;
- A transition relation $\longrightarrow \; \subseteq Q \times L \times Q$;
- An output set O;
- An output function $H : Q \to O$.

A transition system T is said to be:

- *metric*, if the output set O is equipped with a metric $\mathbf{d} : O \times O \to \mathbb{R}_0^+$;
- *countable*, if Q and L are countable sets;
- *finite/symbolic*, if Q and L are finite sets.

We will follow standard practice and denote an element $(q, l, p) \in \; \longrightarrow$ by $q \xrightarrow{l} p$. Transition systems capture dynamics through the transition relation. For any states $q, p \in Q$, $q \xrightarrow{l} p$ simply means that it is possible to evolve from state q to state p under the action labeled by l.

A time–delay system $\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}, f)$ can be represented by means of the following transition system:

$$T(\Sigma) := (Q, q_0, L, \longrightarrow, O, H), \tag{3}$$

where:

- $Q = \mathcal{X}$;
- $q_0 = \xi_0$;
- $L = \mathcal{U}$;
- $q \xrightarrow{u} p$, if $x_\tau(q, u) = p$ for some $\tau \in \mathbb{R}^+$;
- $O = \mathcal{X}$;
- $H = 1_\mathcal{X}$.

Transition system $T(\Sigma)$ is metric when the set $O = \mathcal{X}$ is regarded as being equipped with the metric $\mathbf{d}(p, q) = \|p - q\|_\infty$. Note that the set of states and the set of labels of $T(\Sigma)$ are functional spaces and therefore $T(\Sigma)$ is not symbolic.

In this paper we will show how to construct symbolic models that are approximately equivalent to $T(\Sigma)$ and hence to Σ, in the sense of bisimulation equivalence [8], [9]. Bisimulation relations are standard mechanisms to relate the properties of transition systems. Intuitively, a bisimulation relation between a pair of transition systems T_1 and T_2 is a relation between the corresponding sets of states explaining how a state trajectory s_1 of T_1 can be transformed into a state trajectory s_2 of T_2 and vice versa. While typical bisimulation relations require that s_1 and s_2 are observationally indistinguishable, that is $H_1(s_1) = H_2(s_2)$, we shall relax this by requiring $H_1(s_1)$ to be close to $H_2(s_2)$ where closeness is measured with respect to the metric on the output set. The following notion has been introduced in [6] and in a slightly different formulation in [7].

Definition 3: Let $T_1 = (Q_1, q_1^0, L_1, \xrightarrow[1]{}, O, H_1)$ and $T_2 = (Q_2, q_2^0, L_2, \xrightarrow[2]{}, O, H_2)$ be metric transition systems with the same output set O and metric $\mathbf{d}$, and let $\varepsilon \in \mathbb{R}_0^+$ be a given precision. A relation $R \subseteq Q_1 \times Q_2$ is said to be an *ε–approximate* bisimulation relation between T_1 and T_2, if for any $(q_1, q_2) \in R$:

(i) $\mathbf{d}(H_1(q_1), H_2(q_2)) \leq \varepsilon$;

(ii) $q_1 \xrightarrow[1]{l_1} p_1$ implies existence of $q_2 \xrightarrow[2]{l_2} p_2$ such that $(p_1, p_2) \in R$;

(iii) $q_2 \xrightarrow[2]{l_2} p_2$ implies existence of $q_1 \xrightarrow[1]{l_1} p_1$ such that $(p_1, p_2) \in R$.

Moreover T_1 is said to be *ε–bisimilar* to T_2 if:

(iv) there exists an ε–approximate bisimulation relation R between T_1 and T_2 such that $(q_1^0, q_2^0) \in R$.

IV. APPROXIMATELY BISIMILAR SYMBOLIC MODELS

Since in many real applications controllers are implemented through digital devices, we will focus on time–delay systems with digital controllers, i.e. with piecewise–constant control inputs. In the following we refer to time–delay systems with digital controllers as *digital time–delay systems*.

From now on we suppose that the set U of input values of the considered time–delay system $\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}, f)$ contains the origin and that it is a hyper rectangle of the form:

$$U := [a_1, b_1] \times [a_2, b_2] \times ... \times [a_m, b_m],$$

for some $a_i < b_i, i = 1, 2, ..., m$. Furthermore given $\tau \in \mathbb{R}^+$, we consider the following class of control inputs:

$$\mathcal{U}_\tau := \left\{ \begin{array}{c} u \in \mathcal{U} : \text{the time domain of } u \text{ is } [-r, -r+\tau] \\ \text{and } u(t) = u(-r), t \in [-r, -r+\tau] \end{array} \right\}. \tag{4}$$

Given $k \in \mathbb{R}^n$ we denote by $\mathcal{U}_{k,\tau}$ the class of control inputs obtained by the concatenation of k control inputs in $\mathcal{U}_\tau$. Let us denote by $T_\tau(\Sigma)$ the sub–transition system of $T(\Sigma)$ where only control inputs in $\mathcal{U}_\tau$ are considered. More formally define:

$$T_\tau(\Sigma) := (Q_1, q_1^0, L_1, \xrightarrow[1]{}, O_1, H_1),$$

where:

- $Q_1 = \mathcal{X}$;
- $q_1^0 = \xi_0$;
- $L_1 = \{l_1 \in \mathcal{U}_\tau \mid x_\tau(x, l_1) \text{ is defined for all } x \in \mathcal{X}\}$;
- $q \xrightarrow[1]{l_1} p$, if $x_\tau(q, l_1) = p$;
- $O_1 = \mathcal{X}$;
- $H_1 = 1_\mathcal{X}$.

Transition system $T_\tau(\Sigma)$ can be thought of as a time discretization of $T(\Sigma)$ and hence, of Σ. Transition system $T_\tau(\Sigma)$ is metric when we regard $O_1 = \mathcal{X}$ as being equipped with the metric $\mathbf{d}(p, q) = \|p - q\|_\infty$. Note that analogously to $T(\Sigma)$, transition system $T_\tau(\Sigma)$ is not symbolic. The construction of symbolic models for digital time–delay systems relies upon approximations of the set of reachable states and of the set of input signals. Let $R_\tau(\Sigma) \subseteq \mathcal{X}$ be the set of reachable states of Σ at times $t = 0, \tau, ..., k\tau, ...$, i.e. the collection of all states $x \in \mathcal{X}$ for which there exist $k \in \mathbb{N}$ and a control input $u \in \mathcal{U}_{k,\tau}$ so that $x = x_{k\tau}(\xi_0, u)$. The sets $R_\tau(\Sigma)$ and $\mathcal{U}_\tau$, corresponding to Q_1 and L_1 in $T_\tau(\Sigma)$ are functional spaces and therefore are needed to be approximated, in the sense of the following definition.

Definition 4: Consider a functional space $\mathcal{Y} \subseteq C^0(I, Y)$ with $Y \subseteq \mathbb{R}^n$, $I = [a, b]$, $a, b \in \mathbb{R}$, $a < b$. A map $\mathcal{A} : \mathbb{R}^+ \rightarrow 2^{C^0(I,Y)}$ is a *countable approximation* of $\mathcal{Y}$ if for any desired precision $\lambda \in \mathbb{R}^+$:

(i) $\mathcal{A}(\lambda)$ is a countable set;

(ii) for any $y \in \mathcal{Y}$ there exists $z \in \mathcal{A}(\lambda)$ s.t. $\|y - z\|_\infty \leq \lambda$;

(iii) for any $z \in \mathcal{A}(\lambda)$ there exists $y \in \mathcal{Y}$ s.t. $\|y - z\|_\infty \leq \lambda$.

A countable approximation $\mathcal{A}_\mathcal{U}$ of $\mathcal{U}_\tau$ can be easily obtained by defining for any $\lambda_\mathcal{U} \in \mathbb{R}^+$,

$$\mathcal{A}_\mathcal{U}(\lambda_\mathcal{U}) = \left\{ \begin{array}{c} u \in \mathcal{U}_\tau : u(t) = u(-r) \in [U]_{2\lambda_\mathcal{U}}, \\ t \in [-r, -r+\tau] \end{array} \right\} \tag{5}$$

where $[U]_{2\lambda_\mathcal{U}}$ is defined as in (16). By comparing $\mathcal{U}_\tau$ in (4) and $\mathcal{A}_\mathcal{U}(\lambda_\mathcal{U})$ in (5) it is readily seen that $\mathcal{A}_\mathcal{U}(\lambda_\mathcal{U}) \subset \mathcal{U}_\tau$ for any $\lambda_\mathcal{U} \in \mathbb{R}^+$. Under assumptions on U, the set $\mathcal{A}_\mathcal{U}(\lambda_\mathcal{U})$ is nonempty for any $\lambda_\mathcal{U} \in \mathbb{R}^+$. The definition of countable approximations of the set of reachable states $R_\tau(\Sigma)$ is more involved since $R_\tau(\Sigma)$ is a functional space. Let us assume as a first step existence of a countable approximation $\mathcal{A}_\mathcal{X}$ of $R_\tau(\Sigma)$. (In the further development we will derive conditions ensuring existence and construction of $\mathcal{A}_\mathcal{X}$.)

We now have all the ingredients to define a countable transition system that will approximate $T_\tau(\Sigma)$. Given any $\tau \in \mathbb{R}^+$, $\lambda_\mathcal{X} \in \mathbb{R}^+$ and $\lambda_\mathcal{U} \in \mathbb{R}^+$ define the following transition system:

$$T_{\tau,\lambda_\mathcal{X},\lambda_\mathcal{U}}(\Sigma) := (Q_2, q_2^0, L_2, \xrightarrow[2]{}, O_2, H_2), \tag{6}$$

where:

- $Q_2 = \mathcal{A}_\mathcal{X}(\lambda_\mathcal{X})$;
- $q_2^0 \in Q_2$ so that $\|\xi_0 - q_2^0\|_\infty \leq \lambda_\mathcal{X}$;
- $L_2 = \mathcal{A}_\mathcal{U}(\lambda_\mathcal{U})$;
- $q \xrightarrow[2]{l} p$, if $\|p - x_\tau(q, l)\|_\infty \leq \lambda_\mathcal{X}$;
- $O_2 = \mathcal{X}$;
- $H_2 = \imath : Q_2 \hookrightarrow O_2$.

Parameters $\lambda_\mathcal{X}$ and $\lambda_\mathcal{U}$ can be thought of as quantizations of the set $R_\tau(\Sigma)$ and of the space $\mathcal{U}_\tau$, respectively. By construction, the transition system in (6) is countable. We

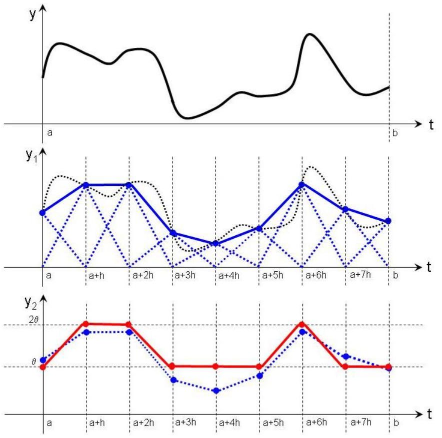

Fig. 1. Spline–based approximation scheme of a functional space.

can now state the following result that relates δ–ISS to the existence of symbolic models for time–delay systems.

Theorem 1: Consider a digital time–delay system $\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}_\tau, f)$ and any desired precision $\varepsilon \in \mathbb{R}^+$. Suppose that Σ is δ–ISS and choose $\tau \in \mathbb{R}^+$ so that $\beta(\varepsilon, \tau) < \varepsilon$. Moreover suppose that there exists a countable approximation $\mathcal{A}_\mathcal{X}$ of $R_\tau(\Sigma)$. Then, for any $\lambda_\mathcal{X} \in \mathbb{R}^+$ and $\lambda_\mathcal{U} \in \mathbb{R}^+$ satisfying the following inequality:

$$\beta(\varepsilon, \tau) + \gamma(\lambda_\mathcal{U}) + \lambda_\mathcal{X} \leq \varepsilon \tag{7}$$

transition systems $T_{\tau,\lambda_\mathcal{X},\lambda_\mathcal{U}}(\Sigma)$ and $T_\tau(\Sigma)$ are ε–bisimilar.

The above result relies upon the existence of a countable approximation for the set of reachable states. In order to address this issue, we consider one possible approximation scheme of functional spaces based on spline analysis [17]. Spline based approximation schemes have been extensively used in the literature of time–delay systems (see e.g. [18] and the references therein).

Let us consider the space $\mathcal{Y} \subseteq C^0(I, Y)$ with $Y \subseteq \mathbb{R}^n$, $I = [a, b]$, $a, b \in \mathbb{R}$ and $a < b$. Given $N \in \mathbb{N}$ consider the following functions (see [17]):

$$s_0(t) = \begin{cases} 1 - (t-a)/h, & t \in [a, a+h], \\ 0, & \textit{otherwise,} \end{cases}$$

$$s_i(t) = \begin{cases} 1 - i + (t-a)/h, & t \in [a + (i-1)h, a + ih], \\ 1 + i - (t-a)/h, & t \in [a + ih, a + (i+1)h], \\ 0, & \textit{otherwise,} \end{cases}$$
$$i = 1, 2, ..., N;$$

$$s_{N+1}(t) = \begin{cases} 1 + (t-b)/h, & t \in [b - r, b], \\ 0, & \textit{otherwise,} \end{cases} \tag{8}$$

where $h = (b-a)/(N+1)$. Functions s_i called *splines*, are used to approximate $\mathcal{Y}$. The approximation scheme that we use is composed of two steps:

(#1) We first approximate a function $y \in \mathcal{Y}$ (Figure 1; upper panel) by means of the piecewise–linear function y_1 (Figure 1; medium panel), obtained by the linear combination of the $N+2$ splines s_i, centered at time $t = a + ih$ with amplitude $y(a + ih)$;

(#2) We then approximate function y_1 by means of function y_2 (Figure 1; lower panel), obtained by the linear combination of the $N+2$ splines s_i, centered at time $t = a + ih$ with amplitude $\tilde{y}_i$ in the lattice[1] $[Y]_{2\theta}$, which minimizes the distance from $y(a + ih)$, i.e.

$$\tilde{y}_i = \arg \min_{y \in [Y]_{2\theta}} \|y - y(a + ih)\|.$$

Given any $N \in \mathbb{N}$, $\theta, M \in \mathbb{R}^+$ let[2]:

$$\Lambda(N, \theta, M) := h^2 M/8 + (N+2)\theta, \tag{9}$$

with $h = (b-a)/(N+1)$. Function Λ will be shown to be an upper bound to the error associated with the approximation scheme that we propose. It is readily seen that for any $\lambda \in \mathbb{R}^+$ and any $M \in \mathbb{R}^+$ there always exist $N \in \mathbb{N}$ and $\theta \in \mathbb{R}^+$ so that $\Lambda(N, \theta, M) \leq \lambda$. Let $N_{\lambda,M}$ and $\theta_{\lambda,M}$ be such that $\Lambda(N_{\lambda,M}, \theta_{\lambda,M}, M) \leq \lambda$. For any $\lambda \in \mathbb{R}^+$ and $M \in \mathbb{R}^+$, define the operator:

$$\psi_{\lambda,M} : \mathcal{Y} \to C^0([a, b]; Y),$$

that associates to any function $y \in \mathcal{Y}$ the function:

$$\psi_{\lambda,M}(y)(t) := \sum_{i=0}^{N_{\lambda,M}+1} \tilde{y}_i s_i(t), \quad t \in [a, b], \tag{10}$$

where $\tilde{y}_i \in [Y]_{2\theta_{\lambda,M}}$ and $\|\tilde{y}_i - y(a + ih)\| \leq \theta_{\lambda,M}$, for any $i = 0, 1, ..., N_{\lambda,M} + 1$. Note that the operator $\psi_{\lambda,M}$ is not uniquely defined. For any given $M \in \mathbb{R}^+$ and any given precision $\lambda \in \mathbb{R}^+$ define:

$$\mathcal{A}_{\mathcal{Y},M}(\lambda) := \psi_{\lambda,M}(\mathcal{Y}). \tag{11}$$

The above approximation scheme is employed to construct countable approximations of the set $R_\tau(\Sigma)$ of reachable states (see Proposition 1). Consider a digital time–delay system $\Sigma = (X, \mathcal{X}, \xi_0, U, \mathcal{U}_\tau, f)$ and suppose that:

(A.1) Σ is δ–ISS;

(A.2) X and U are bounded sets;

(A.3) Functional f is Frechét differentiable in $C^0([-\Delta, 0]; \mathbb{R}^n) \times \mathbb{R}^m$;

(A.4) The Frechét differential $J(\phi, u)$ of f is bounded on bounded subsets of $C^0([-\Delta, 0]; \mathbb{R}^n) \times \mathbb{R}^m$.

Under the above assumptions, the following bounds are well defined:

$$\begin{aligned} B_X &= \sup_{x \in X} \|x\|, \\ B_U &= \sup_{u \in U} \|u\|, \\ B_J &= \sup_{(\phi,u) \in S} \|J(\phi, u)\|, \\ M &= (\beta(B_X, 0) + \gamma(B_U) + B_U)\kappa B_J, \end{aligned} \tag{12}$$

where

$$S = \{(\phi, u) \in C^0([-\Delta, 0]; X) \times U : \|\phi\|_\infty \leq B_X, \|u\| \leq B_U\},$$

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[1] We recall that the set $[Y]_{2\theta}$ is defined as in (16).

[2] The real M is a parameter associated with $\mathcal{Y}$ and its role will become clear in the subsequent developments.

and κ is the Lipschitz constant of functional f in the bounded set S and $\|J(\phi,u)\|$ denotes the norm of the operator $J(\phi,u): C^0([-\Delta,0];\mathbb{R}^n)\times\mathbb{R}^m \rightarrow \mathbb{R}^n$. We can now give the following result that points out sufficient conditions for the existence of countable approximations of $R_\tau(\Sigma)$.

Proposition 1: Consider a digital time–delay system $\Sigma=(X,\mathcal{X},\xi_0,U,\mathcal{U}_\tau,f)$, satisfying assumptions (A.1-4) and the following conditions:

(A.5)

$$\begin{array}{ll} \xi_0 \in PC^2([-\Delta,0];X), & \|\xi_0\|_\infty \leq B_X^0 \leq B_X, \\ \left\|D^2\xi_0\right\|_\infty < M, & \beta(B_X^0,0)+\gamma(B_U) \leq B_X, \\ \beta(B_X^0,\tau)+\gamma(B_U) \leq B_X^0, & \tau > 2\Delta, \end{array}$$

with M as in (12). Then the set $\mathcal{A}_\mathcal{X}$ defined for any $\lambda_\mathcal{X} \in \mathbb{R}^+$ by:

$$\mathcal{A}_\mathcal{X}(\lambda_\mathcal{X}) = \psi_{\lambda_\mathcal{X},M}(R_\tau(\Sigma)), \tag{13}$$

with $\psi_{\lambda_\mathcal{X},M}$ as in (10), is a countable approximation of $R_\tau(\Sigma)$.

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input:
time–delay system $\Sigma=(X,\mathcal{X},\xi_0,U,\mathcal{U},f)$ satisfying
assumptions (A.1-5);
parameters $\tau,N,\theta,\lambda_\mathcal{U},M$;
init:
$k:=0$;
$Q^k:=\{q_2^0\}$, where $q_2^0=\psi_{\lambda,M}(\xi_0)$, with $\psi_{\lambda,M}$ defined
as in (10) and $\lambda=\Lambda(N,\theta,M)$;
$Q^{k-1}:=\emptyset$;
$\xrightarrow[k]{} \;:=\emptyset$;
$H_2:=\imath: Q_2 \hookrightarrow O_2$;
$h:=\Delta/(N+1)$;
while $Q^k \neq Q^{k-1}$ do
    foreach $q \in Q^k$ do
        foreach $l_2 \in [U]_{2\lambda_\mathcal{U}}$ do
            compute $z:=x_\tau(q,l_2)$;
            compute $p=\psi_{\lambda,M}(z)$, with $\psi_{\lambda,M}$ defined
            as in (10) and $\lambda=\Lambda(N,\theta,M)$;
            $Q^{k+1}:=Q^k \cup \{p\}$;
            $\xrightarrow[k+1]{} \;:=\; \xrightarrow[k]{} \cup \{(q,l_2,p)\}$;
        end
    end
    k:=k+1;
end
output: $T_{\tau,N,\theta,\lambda_\mathcal{U}}(\Sigma):=(Q^k,q_2^0,[U]_{\lambda_\mathcal{U}},\xrightarrow[k]{},\mathcal{X},H_2)$
```

Algorithm 1: Construction of symbolic models for time–delay systems.

We now have all the ingredients to define a symbolic model for digital time–delay systems. Given $\tau \in \mathbb{R}^+$, $\theta,\lambda_\mathcal{U} \in \mathbb{R}^+$ and $N \in \mathbb{N}$, consider the transition system

$$T_{\tau,N,\theta,\lambda_\mathcal{U}}(\Sigma):=(Q_2,q_2^0,L_2,\xrightarrow[2]{},O_2,H_2), \tag{14}$$

where:

- $Q_2=\mathcal{A}_\mathcal{X}(\Lambda(N,\theta,M))$ with $\mathcal{A}_\mathcal{X}$ as in (13) with $\lambda_\mathcal{X}=\Lambda(N,\theta,M)$ and M as in (12);
- $q_2^0 \in Q_2$, is such that $\|q_2^0-\xi_0\|_\infty \leq \Lambda(N,\theta,M)$;
- $L_2=\mathcal{A}_\mathcal{U}(\lambda_\mathcal{U})$;
- $q \xrightarrow[2]{l} p$, if $\|p-x_\tau(q,l)\|_\infty \leq \Lambda(N,\theta,M)$;
- $O_2=\mathcal{X}$;
- $H_2=\imath: Q_2 \hookrightarrow O_2$.

Note that the transition system in (14) coincides with the one in (6) with $\lambda_\mathcal{X}=\Lambda(N,\theta,M)$. It is readily seen that:

Proposition 2: If the time–delay system Σ satisfies assumptions (A.1-5), transition system $T_{\tau,N,\theta,\lambda_\mathcal{U}}(\Sigma)$ in (14) is symbolic.

Transition system $T_{\tau,N,\theta,\lambda_\mathcal{U}}(\Sigma)$ can be constructed by analytical and/or numerical integration of the solutions of the time–delay system. One possible construction scheme is illustrated in Algorithm 1, which proceeds, as follows. The set Q^k of states of the symbolic model at step $k=0$ is initialized to contain the (only) symbol $q_2^0=\psi_{\lambda,M}(\xi_0)$ that is associated with the initial condition ξ_0. Then, for any initial condition $q \in Q^k$ and any control input $l_2 \in [U]_{2\lambda_\mathcal{U}}$, the algorithm computes the solution $z=x_\tau(q,l_2)$ of the differential equation in (1) at time $t=\tau$, and it adds the symbol $p=\psi_{\lambda,M}(z)$ to Q^k. In the end of this basic step, index k is increased to $k+1$ and the above basic step is repeated. The algorithm continues by adding symbols to Q^k since no more symbols are found, or equivalently, since a step k^* is found, for which $Q^{k^*}=Q^{k^*+1}$. Termination properties of the proposed algorithm are discussed in the following result.

Theorem 2: Algorithm 1 terminates in a finite number of steps.

We can now give the main result of this paper.

Theorem 3: Consider a digital time–delay system $\Sigma=(X,\mathcal{X},\xi_0,U,\mathcal{U}_\tau,f)$ and any desired precision $\varepsilon \in \mathbb{R}^+$. Suppose that assumptions (A.1-5) are satisfied. Moreover let $\tau,\theta,\lambda_\mathcal{U} \in \mathbb{R}^+$ and $N \in \mathbb{N}$ satisfy the following inequality

$$\beta(\varepsilon,\tau)+\gamma(\lambda_\mathcal{U})+\Lambda(N,\theta,M) \leq \varepsilon, \tag{15}$$

with Λ as in (9) and M as in (12). Then transition systems $T_\tau(\Sigma)$ and $T_{\tau,N,\theta,\lambda_\mathcal{U}}(\Sigma)$ are ε–bisimilar.

Proof: The set $\mathcal{A}_\mathcal{U}$ is a countable approximation of U and by Proposition 1, $\mathcal{A}_\mathcal{X}$ is a countable approximation of $R_\tau(\Sigma)$. Choose $\lambda_\mathcal{X} \in \mathbb{R}^+$ and $\lambda_\mathcal{U} \in \mathbb{R}^+$ satisfying inequality (7). There exist $\theta \in \mathbb{R}^+$ and $N \in \mathbb{N}$ so that $\lambda_\mathcal{X}=\Lambda(N,\theta,M)$ and hence inequality (15) holds. Finally the result holds as a direct application of Theorem 1. ■

Since by the above result a symbolic model can be constructed which is approximately bisimilar to δ–ISS nonlinear time–delay systems, control design of nonlinear time–delay systems can be translated to control design of symbolic models, for which there exists a wealth of results in the computer science literature, as for example supervisory control [4] and algorithmic game theory [5].

V. DISCUSSION

In this paper we showed that incrementally input–to–state stable digital time–delay systems admit symbolic models that are approximately bisimilar to the original system, with a precision that can be rendered as small as desired. We also presented an algorithm for the computation of the proposed symbolic models. Convergence of the algorithm in finite time is ensured under a boundness assumption on the state and input spaces.

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VI. APPENDIX

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the sets of natural, integer, real, positive and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$ the i–th element of x is denoted by x_i; furthermore $\|x\|$ denotes the infinity norm of x; we recall that $\|x\| := max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ is the absolute value of x_i. For any $A \subseteq \mathbb{R}^n$ and $\theta \in \mathbb{R}^+$ define

$$[A]_\theta := \{a \in A \; | a_i = k_i \theta, \;\; k_i \in \mathbb{Z}, i = 1, ..., n\}. \qquad (16)$$

Given a measurable and locally essentally bounded function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$, the (essential) supremum norm of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (ess)sup\{\|f(t)\|, t \geq 0\}$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau[$, and $f(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. Given $k, n \in \mathbb{N}$ with $n \geq 1$ and $I = [a, b] \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$ let $C^k(I; \mathbb{R}^n)$ be the space of functions $f : I \rightarrow \mathbb{R}^n$ that are continuously differentiable k times. Given $k \geq 1$, let $PC^k(I; \mathbb{R}^n)$ be the space of $C^{k-1}(I; \mathbb{R}^n)$ functions $f : I \rightarrow \mathbb{R}^n$ whose k–th derivative exists except in a finite number of reals, and it is bounded, i.e. there exist $\gamma_0, \gamma_1, ..., \gamma_s \in \mathbb{R}^+$ with $a = \gamma_0 < \gamma_1 < ... < \gamma_s = b$ so that $D^k f$ is defined on each open interval (γ_i, γ_{i+1}), $i = 0, 1, ..., s-1$ and $\max_{i=0,1,...,s-1} \sup_{t \in (\gamma_i, \gamma_{i+1})} \|D^k f(t)\|_\infty < \infty$. For any continuous function $x(s)$, defined on $-\Delta \leq s < a$, $a > 0$, and any fixed t, $0 \leq t < a$, the standard symbol x_t will denote the element of $C^0([-\Delta, 0]; \mathbb{R}^n)$ defined by $x_t(\theta) = x(t + \theta)$, $-\Delta \leq \theta \leq 0$. The identity map on a set A is denoted by 1_A. Given two sets A and B, if A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \rightarrow B$ the symbol $f(A)$ denotes the image of A through f, i.e. $f(A) := \{b \in B : \exists a \in A$ s.t. $b = f(a)\}$.