Positive Switching Systems

Elena De Santis, Giordano Pola

Department of Electrical Engineering and Computer Science, Center of Excellence DEWS, University of L'Aquila, 67040 Poggio di Roio, L'Aquila, Italy
desantis,pola@ing.univaq.it

Abstract. In this paper we study switching systems, a subclass of hybrid systems, and we address different positivity questions: we analyze systems where state variables evolve while remaining non–negative independently from the control chosen and external events actions (systems in this class are usually called *positive systems*), and systems where a control strategy can be designed in order to maintain the state evolution non–negative, against external events acting on the system (we call systems in this class *controlled positive*).

1 Introduction

A hybrid system consists of a finite family of continuous–time or discrete–time subsystems and a rule that orchestrates the switching among them. We consider the subclass of *linear switching systems* [3], where discrete transitions are caused by a discrete input that acts as an external and unknown disturbance and where continuous dynamics are given by linear dynamical control systems. The motivation for considering this particular subclass is that switching systems are an appropriate abstraction for modelling important complex systems such as air traffic management systems (e.g. [4]) or automotive engines (e.g. [5]). This class has moreover remarkable potential in modelling systems coming from many application domains, including management, economics, communication and biological systems, since it offers the framework to deal with uncertainties, as well as with non-smooth phenomena.

In most cases positivity of the variables is a consequence of the nature of the phenomenon itself. For this reason in this paper we deal with switching systems, where positivity is addressed from different perspectives: in fact we analyze systems where state variables evolve while remaining non–negative independently from the control chosen and from external events actions (systems in this class are usually called *positive systems*), and systems where a control strategy can be designed in order to maintain the state evolution non–negative, against external events acting on the system (we call systems in this class *controlled positive*).

We generalize to the switching system framework definitions and results previously stated for classical dynamical systems (see [6] for positive systems and [1] for controlled positive systems). Some of the tools that we use, are derived from [3], where problems with safety specifications were studied in a framework comprising the present one.

The paper is organized as follows. In Section 2 we give preliminary definitions. In Section 3 we characterize positive switching systems. In Section 4 controlled positivity of switching systems is addressed. Finally concluding remarks are offered in Section 5.

Notation. R, N denotes respectively the sets of reals and of integers. The symbol R^n_+ denotes the non–negative orthant of the Euclidean space R^n. If x, y are vectors of R^n, then $x \leq y$ holds componentwise. The symbol $\mathbf{0}_n$ denotes the zero vector of the Euclidean space R^n. Given a matrix $A \in R^{n\times m}$, $Im(A)$ denotes the range space of A and $A \geq 0$ means that each entry of A is non–negative. Given a linear subspace F of R^n, the symbol $F^\perp$ denotes the orthogonal complement of F in R^n. The identity matrix in R^n is denoted by $\mathbf{I}_n$. The symbol $card(\Omega)$ denotes the cardinality of a given set Ω. Given $a, b \in N$, $a \leq b$ the symbol $[a, b]$ denotes the set $\{t \in N : a \leq t \leq b\}$.

2 Preliminary Definitions

In this section, we formally introduce the class of *discrete–time linear switching systems* that we consider in this paper, following the general model of hybrid automata introduced in [7].

The hybrid state ξ of a linear switching system is composed of two components: the discrete state q_i, belonging to a finite set Q and the continuous state x, belonging to a linear space R^{n_i}, whose dimension n_i depends on q_i. The evolution of the discrete state is governed by a Finite State Machine (FSM); a transition $e = (q_i, \sigma, q_h)$ may occur at time t from the discrete state q_i to the discrete state q_h, if the discrete disturbance σ occurs at time t. The evolution of the continuous state is described by a set of discrete–time linear dynamical control systems, whose matrices depend on the current discrete state. Whenever a transition e occurs, the continuous state x is instantly reset to a new value $M(e)x$, where $M(e)$ is a matrix depending on the transition e. More formally,

Definition 1. *A linear switching system $\mathcal{S}$ is a tuple (Ξ, Θ, S, E, M) where:*

- $\Xi = \bigcup_{q_i \in Q} \{q_i\} \times R^{n_i}$ *is the hybrid state space, where:*
  ◦ $Q = \{q_i, i \in J\}$ *is the finite set of discrete states,* $J \subset N$*;*
  ◦ R^{n_i} *is the continuous state space associated with* $q_i \in Q$*;*
- $\Theta = \Sigma \times R^m$ *is the hybrid input space, where:*
  ◦ Σ *is the finite set of discrete disturbances;*

∘ R^m *is the continuous input space;*

- S *is a map associating to any discrete state* $q_i \in Q$ *the following discrete–time linear dynamical control system:*

$$x(t+1) = A_i x(t) + B_i u(t),$$

where $x(t) \in R^{n_i}$ *is the continuous state and* $u(t) \in R^m$ *is the continuous input function at time* $t \in N$*;*
- $E \subset Q \times \Sigma \times Q$ *is a collection of transitions;*
- M *is a function that associates to any* $e = (q_i, \sigma, q_h) \in E$ *the reset matrix* $M(e) \in R^{n_h \times n_i}$.

We now formally define the semantics of linear switching systems. We assume throughout the paper that *the discrete disturbance is not available for measurements*, thus yielding a non–deterministic system. As defined in [7], a *hybrid time basis* τ is an infinite or finite sequence of sets $I_j = [t_j, t'_j]$, with $t'_j = t_{j+1}$; set $card(\tau) = L + 1$. If $L < \infty$, then t'_L can be finite or infinite. Since linear switching systems are time invariant, we assume without loss of generality that $t_0 = 0$ in any hybrid time basis. Given a hybrid time basis τ, any time instant t'_j is called *switching time*. Throughout the paper, we assume that there is a minimum time separation between two consecutive switching times:

Assumption 1 $t'_j - t_j \geq 1$, for any $j = 0, 1, ..., L$ and any hybrid time basis τ.

Denote by $\mathcal{T}$ the set of all hybrid time bases satisfying Assumption 1. The temporal evolution of a linear switching system can be now defined as follows.

Definition 2. *(Switching system execution) An execution* χ *of a linear switching system* $\mathcal{S}$ *is a collection* $(\xi_0, \tau, \sigma, u, \xi)$ *with* $\xi_0 \in \Xi$, $\tau \in \mathcal{T}$, $\sigma : N \to \Sigma$, $u : N \to R^m$, $\xi : N \times N \to \Xi$. *The hybrid state evolution* ξ *is defined as follows:*

$$\begin{array}{l} \xi(0,0) = \xi_0, \\ \xi(t,j) = (q(j), x(t,j)), t \in I_j, j = 0,1,...,L, \\ \xi(t_{j+1}, j+1) = (q(j+1), M(e_j)x(t'_j, j)), \\ j = 0,1,...,L, \end{array}$$

where $q : N \to Q$, $e_j = (q(j), \sigma(j), q(j+1)) \in E$ *and* $x(t,j)$ *is the solution at time* t *of the dynamical system* $S(q(j))$*, with initial time* t_j*, initial condition* $x(t_j, j)$ *and continuous input* u*.*

Given $\mathcal{S}$ and an execution χ, set $\eta(t) = \xi(t,j), t \in [t_j, t'_j - 1], j = 0, 1, ..., L$. We assume that $\eta|_{[0,t]}$ *is available for control synthesis* at time t and therefore the set

$$\mathcal{Y} = \{\eta|_{[0,t]}, \eta : N \to \Xi, t \geq 0\}$$

embeds all the information on the hybrid state evolution available for control purposes. A *control strategy* φ is a function $\varphi : \mathcal{Y} \to R^m$. A switching system

$\mathcal{S}$ together with a control strategy φ is called *controlled switching system* and its executions with $u(t) = \varphi(\eta|_{[0,t]})$, $t \geq 0$ are called *controlled executions.*

Define the *hybrid non–negative orthant*, as follows:

$$\Pi = \bigcup_{i \in J} \{q_i\} \times R_+^{n_i} \subset \Xi,$$

and consider the hybrid state constraint:

$$\xi(t,j) \in \Pi, \forall t \in I_j, j = 0, 1, ..., L-1. \tag{1}$$

In the following sections we study conditions for which hybrid constraint (1) can be fulfilled by the hybrid state evolution of a given switching system.

We conclude by introducing some notations that will be useful in the next sections. Given a switching system $\mathcal{S}$, for any given $i \in J$, let E_i be the set of transitions starting from state q_i, i.e.

$$E_i = \left\{ e^i = (q_i, \sigma, q_h), h : e^i \in E \right\},$$

and set $card\,(E_i) = \mu_i$. Define the set of matrices $V_i \in R^{\nu_i \times n_i}$, $i \in J$, such that $V_i = \mathbf{I}_{n_i}$ if $\mu_i = 0$ and

$$V_i = \begin{pmatrix} \mathbf{I}_{n_i} \\ M\left(e_1^i\right) \\ \vdots \\ M\left(e_{\mu_i}^i\right) \end{pmatrix}, e_k^i \in E_i, k = 1, 2, ..., \mu_i,$$

otherwise. By definition of V_i, the value of the integer ν_i can be easily computed.

3 Positive Switching Systems

In this section we study conditions for which hybrid constraints (1) can be fulfilled by the hybrid state evolution of a given switching system for any control strategy taking nonnegative values and for any initial hybrid state in Π.

By generalizing to switching systems the definition of positive systems, as introduced by Luenberger in [6], the following definition is obtained.

Definition 3. *A switching system $\mathcal{S}$ is positive if for any control strategy $\varphi^+ : \mathcal{Y} \rightarrow R_+^m$ and for any controlled execution with initial hybrid state in Π, hybrid state constraint (1) is satisfied.*

We can state the following result that gives a necessary and sufficient condition for a switching system to be positive.

Theorem 1. *A switching system $\mathcal{S}$ is positive if and only if* $\begin{pmatrix} V_iA_i \; V_iB_i \end{pmatrix} \geq 0$*, for any* $i \in J$*.*

Proof. If starting at $t_0 = 0$ from $\xi_0 = (q_i, x_0) \in \Pi$ a switching never occurs, condition in Definition 3 is satisfied if and only if $\begin{pmatrix} A_i \; B_i \end{pmatrix} \geq 0$. If a switching occurs after one step of time, the state just after the switching (i.e. at $t = 1$, $j = 1$) belongs to Π, for any control strategy φ^+, if and only if $\begin{pmatrix} V_iA_i \; V_iB_i \end{pmatrix} \geq 0$ for any $i \in J$. Therefore the condition in the statement is necessary. As for the sufficiency, assume $\xi(t_j, j) \in \Pi$. If a switching occurs after t steps of time then, at any time before the switching, the state belongs to Π, since $\begin{pmatrix} A_i \; B_i \end{pmatrix} \geq 0$, for any $i \in J$. Just after the transition $e = (q_i, \sigma, q_h) \in E_i$, since the condition $\begin{pmatrix} V_iA_i \; V_iB_i \end{pmatrix} \geq 0$ for any $i \in J$, implies that the matrix $M(e) \begin{pmatrix} A_i^k \; A_i^{k-1}B_i \; \ldots \; A_iB_i \; B_i \end{pmatrix}$ is positive, then $\xi(t_{j+1}, j+1) \in \Pi$. Therefore by induction the result follows.

4 Controlled Positive Switching Systems

In this section we study conditions for which it is possible to design a suitable control strategy such that the hybrid state constraint (1) can be fulfilled by the controlled hybrid state evolution of a given switching system for any initial hybrid state in Π.

Results of this section have been obtained by generalizing definitions and results of [1] to the case of switching systems.

Definition 4. *A switching system $\mathcal{S}$ is controlled positive if there exists a control strategy such that for any controlled execution with initial hybrid state in Π, hybrid state constraint (1) is satisfied.*

The notion of controlled positivity for switching systems is closely related to the notion of controlled invariance as studied in [3] and therefore a geometrical characterization of controlled positive switching systems can be derived by results established in [3]. More formally:

Definition 5. *A set* $\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi$ *is controlled invariant for a switching system $\mathcal{S}$ if there exists a control strategy such that for any controlled execution with initial hybrid state* $\xi_0 \in \Omega$*,*

$$\xi(t, j) \in \Omega, \forall t \in I_j, \forall j = 0, 1, ..., L-1.$$

The following result gives a necessary and sufficient condition for a set to be controlled invariant.

Lemma 1. *A set* $\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i$ *is controlled invariant for a switching system $\mathcal{S}$ if and only if for any* $i \in J$ *and for any* $x \in \Omega_i$ *there exists* $u \in R^m$ *such that:*

(i) $A_i x + B_i u \in \Omega_i$*;*
(ii) $M(e)\,(A_i x + B_i u) \in \Omega_h$*, for any* $e = (q_i, \sigma, q_h) \in E_i$*.*

Proof. Necessity: since starting from any initial state in Ω a switching could never occur, then each set Ω_i has to be controlled invariant for the system $S(q_i)$. Therefore condition (i) is necessary. A switching can occur after one step of time, and hence condition (ii) is necessary. Sufficiency: if starting at time t_j from (q_i, x), $x \in \Omega_i$, a switching never occurs, then condition (i) implies that the state evolution remains in Ω. Suppose now that a switching occurs at time $t'_j \geq t_j + 1$. Conditions (i) and (ii) imply that for any $x(t_j, j) \in \Omega_i$ there exists an input $u \in R^m$ such that $x(t_j + 1, j) \in \Omega_i \cap \Lambda_i$, where $\Lambda_i = \bigcap_{e=(q_i,\sigma,q_h)\in E} (M(e))^{-1}\,\Omega_h$. Since $\Omega_i \cap \Lambda_i \subset \Omega_i$, then for any $x(t_j, j) \in \Omega_i \cap \Lambda_i$ there exists an input $u \in R^m$ such that $x(t_j + 1, j) \in \Omega_i \cap \Lambda_i$, and therefore $\Omega_i \cap \Lambda_i$ is controlled invariant for dynamical system $S(q_i)$. By definition of Λ_i, there exists a control strategy such that $x(t_{j+1}, j+1) \in \Omega_h$, for any switching time $t'_j \geq t_j + 1$, for any $e = (q_i, \sigma, q_h) \in E_i$. Therefore, by induction, the set Ω is controlled invariant.

By Definition 5, a switching system $\mathcal{S}$ is controlled positive if and only if Π is a controlled invariant set for $\mathcal{S}$ and therefore by specializing Lemma 1 to the hybrid non–negative orthant Π, the following result is obtained.

Theorem 2. *A switching system* $\mathcal{S}$ *is controlled positive if and only if for any* $i \in J$ *the column vectors of the matrix* $V_i A_i$ *belong to the set* $Im\,(V_i B_i) + R_+^{\nu_i}$*.*

Proof. From Lemma 1, $\mathcal{S}$ is controlled positive if and only if $\forall i \in J, \forall x \geq \mathbf{0}_{n_i}, \exists u \in R^m$ such that $A_i x + B_i u \geq \mathbf{0}_{n_i}$ and $M(e)\,(A_i x + B_i u) \geq \mathbf{0}_{n_k}$, for any $e = (q_i, \sigma, q_k) \in E_i$, that is equivalent to:

$$\forall i \in J, \forall z \in Z_i, \exists u \in R^m : V_i A_i z + V_i B_i u \geq \mathbf{0}_{\nu_i}, \tag{2}$$

where Z_i is the set of unitary orthonormal versors of R^{n_i}. Let us denote by m_{hi} the $h-th$ column vector of $V_i A_i$. Then condition (2) is equivalent to:

$$\forall i \in J, \forall h = 1, 2, ..., n_i, \exists u \in R^m : m_{hi} + V_i B_i u \geq \mathbf{0}_{\nu_i}.$$

Since this last condition holds if and only if $m_{hi} \in Im\,(V_i B_i) + R_+^{\nu_i}$, $\forall h = 1, 2, ..., n_i$, then the result follows.

Recall that a subset Ω of a linear space R^n is said to be a *convex cone* (for brevity cone) if it is closed under conical combinations. This is equivalent to say $\alpha y + \beta z \in \Omega$ whenever $y, z \in \Omega$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. A convex subset of a convex set is called a *face* of the set if, whenever a segment, contained in the set, has a relative internal point in common with the subset, the extreme points of the segment belong to the convex subset.

A *ray* (recall that a ray is the set of all non–negative multiples of a non zero vector) is said to be an *extreme ray* for a cone if it is a face of the cone. A cone Ω is *pointed* if the maximal linear subspace contained in Ω is $\{\mathbf{0}_n\}$. We define the set of *generators* of a pointed polyhedral cone any set obtained by choosing a non zero vector from each extreme ray of the cone. Finally, any pointed polyhedral cone is the conical extension of a set of generators of the cone itself.

By definitions introduced above, the set $Im\,(V_iB_i) + R_+^{\nu_i}$ is a polyhedral cone and the set $Im\,(V_iB_i)^\perp \cap R_+^{\nu_i}$, $i \in J$, is a pointed polyhedral cone. Therefore, the geometrical condition stated in Theorem 2 can be equivalently rewritten as:

Theorem 3. *A switching system $\mathcal{S}$ is controlled positive if and only if for any $i \in J$*

$$G_iV_iA_i \geq 0,$$

where the rows of G_i are the generators of $Im\,(V_iB_i)^\perp \cap R_+^{\nu_i}$.

The proof is a direct consequence of results established in [1], and therefore it can be omitted.

Remark 1. In [2] the intersection between a linear subspace and the non–negative orthant is extensively studied and an efficient algorithm is developed for the computation of its generators.

We conclude this section, by showing that positivity of a switching system can be obtained by means of a linear state feedback control strategy. More formally, a control strategy φ is said to be a *static hybrid linear state feedback*, if for any $q_i \in Q$ there exists a matrix $K_i \in R^{m\times n_i}$ such that:

$$\begin{aligned}&\varphi(\eta|_{[0,t]}) = K_ix(t,j),\\ &\eta\,(t) = (q_i, x\,(t,j)).\end{aligned}$$

A switching system $\mathcal{S}$ is said to be *controlled positive via static hybrid linear state feedback* if it is controlled positive by means of a control strategy that is a static hybrid linear state feedback.

We can give now the following result.

Theorem 4. *A switching system $\mathcal{S}$ is controlled positive if and only if it is controlled positive via static hybrid linear state feedback.*

Proof. The sufficiency is obvious. As for the necessity, consider the condition (2) and let Z_i be the set of unitary orthonormal versors of R^{n_i}, and for any $z_h \in Z_i$, let u_h be a vector such that $V_iA_iz_h + V_iB_iu_h \geq \mathbf{0}_{\nu_i}$. Since any $x \in R_+^{n_i}$ can be rewritten as:

$$x = \sum\nolimits_{h=1}^{n_i} z_h\alpha_h = \begin{pmatrix}\alpha_1\ \alpha_2\ \ldots\ \alpha_{n_i}\end{pmatrix}',$$

with $\alpha_h \geq 0$, then by linearity, $V_iA_ix + V_iB_iu \geq \mathbf{0}_{\nu_i}$, with $u = \sum_{h=1}^{n_i} u_h\alpha_h$. Finally by setting $u = K_ix$, $K_i = \begin{pmatrix}u_1\ u_2\ \ldots\ u_{n_i}\end{pmatrix}$, the result follows.

5 Concluding Remarks

In this paper we studied positive switching systems, where state variables evolve while remaining non–negative independently from the control chosen and from external events actions, and controlled positive switching systems, where a control strategy can be designed in order to maintain the state evolution non–negative, against external events acting on the system.

Further work will concentrate to the generalization of the positivity notions explored in this paper for the class of discrete–time linear switching systems, to more general classes of hybrid systems, whose semantic is indeed a powerful tool for modeling diverse non–smooth phenomena arising in many application domains of interest.

Acknowledgements: This work has been partially supported by the HYCON Network of Excellence, contract number FP6-IST-511368 and by Ministero dell'Istruzione dell'Università e della Ricerca under Project MACSI (PRIN 05).

References

1. d'Alessandro P, De Santis E (1994) Positiveness of dynamic systems with non positive coefficient matrices. IEEE Trans. on Automatic Control 39(1):131–134
2. d'Alessandro P (1997) A conical approach to linear programming, scalar and vector optimization problems. Gordon and Breach Science Publishers
3. De Santis E, Di Benedetto M D, Berardi L (2004) Computation of maximal safe sets for switching linear systems. IEEE Trans. on Automatic Control 49(2):184–195
4. De Santis E, Di Benedetto M D, Di Gennaro S, D'Innocenzo A, Pola G (2006) Critical Observability of a Class of Hybrid Systems and Application to Air Traffic Management. Lecture Notes in Control and Information Sciences – Springer Verlag. To appear
5. De Santis E, Di Benedetto M D, Pola G (2006) Digital Idle Speed Control of Automotive Engines: A Safety Problem for Hybrid Systems. Nonlinear Analysis, 6–th Special Issue on Nonlinear Analysis: Hybrid Systems and Applications. To appear
6. Luenberger D G (1979) Introduction to dynamic systems. J. Wiley & Sons New York
7. Lygeros J, Tomlin C, Sastry S (1999) Controllers for reachability specifications for hybrid systems, Automatica, Special Issue on Hybrid Systems, 35:349–370
8. Tomlin C, Lygeros J, Sastry, S (1998) Synthesizing Controllers for Nonlinear Hybrid Systems. Hybrid Systems: Computation and Control, S. Sastry and T. Henzinger Eds. Lecture Notes in Computer Science, 1386:360–373