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Stabilizability of linear switching systems[☆]

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Abstract

A complete characterization of stabilizability for linear switching systems is not available in the literature. In this paper, we show that the asymptotic stabilizability of linear switching systems is equivalent to the existence of a hybrid Lyapunov function for the controlled system, for a suitable control strategy. Further, we prove that asymptotic stabilizability of a switching system with minimum dwell time, is equivalent to Input to State Stability (ISS) of the controlled switching system, with a stabilizing control law. We then derive some structural reductions of the hybrid state space, which allow a decomposition of the original problem into simpler subproblems. The relationships between this approach and the well-known Kalman decomposition of linear dynamic control systems are explored.

Keywords: Linear switching systems; Stabilizability; Lyapunov functions; Kalman decomposition; Hybrid systems

1. Introduction

Stability properties for hybrid systems have been extensively investigated for the past few years (e.g. see [3,20,12], and [18]). However, checking stabilizability of hybrid systems is not an easy task in general (see e.g. [12]).

In this paper, we derive stabilizability conditions for linear switching systems, a particular class of hybrid systems where the continuous dynamics and the reset functions are linear and the transitions depend only on an event that acts as a discrete disturbance. The continuous dynamics are given by a linear control system (whose dynamic matrices depend on the current discrete state), where the input function can be designed for controlling purposes. This class is a generalization of the class of *linear polysystems*, as defined in [4].

We then introduce the notion of *Hybrid Lyapunov function* and show that Uniform Asymptotic Stabilizability (*UAS*) of linear switching systems having a minimum dwell time is equivalent to the existence of a hybrid Lyapunov function for the controlled system, for a suitable control strategy. Moreover, we prove that *UAS* is equivalent to Input to State Stability (shortly *ISS*, as introduced in [17]), of the controlled switching system, with a suitable stabilizing control law. For the controlled switching system, which is in general *nonlinear*, these results are summarized in the

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following table:

$$\boxed{\begin{array}{c}\exists\ \textit{Hybrid Lyapunov}\\ \textit{Function}\end{array}} \iff \boxed{\textit{UAS}} \iff \boxed{\textit{ISS}}$$

Under the assumption of linearity of the continuous dynamics, the results we present in this paper are stronger than the results presented in the literature. In particular, in [11] Liberzon argued that a suitable common Lyapunov function for a nonlinear switching system, with identity reset and arbitrary switchings, implies *ISS* of the same system. Conversely, for the same class of switching systems, Mancilla et al. in [14] showed that *ISS* implies the existence of a common *ISS* Lyapunov function. Again for the class of nonlinear switching systems, with identity reset and arbitrary switchings, Uniform Global Asymptotic Stability (*UGAS*) was proved to be equivalent to the existence of a Lyapunov function in [4]. Then, since by definition *ISS* implies *UGAS*, for the class of nonlinear switching systems with identity reset and arbitrary switchings, the following implications follow from the results presented in the papers mentioned above:

$$\boxed{\begin{array}{c}\exists\ \textit{Lyapunov}\\ \textit{Function}\end{array}} \underset{[4]}{\iff} \boxed{\textit{UGAS}} \Leftarrow \boxed{\textit{ISS}} \underset{[11],\,[14]}{\iff} \boxed{\begin{array}{c}\exists\ \textit{ISS}\\ \textit{Lyapunov}\\ \textit{Function}\end{array}}$$

In this paper, we also present structural reductions of the hybrid state space, which can be used to decompose the original problem of checking stabilizability into simpler subproblems. To do so, we introduce the notion of hybrid invariant subspace. Based on this notion and on [8], we show that the given system can be decoupled into controlled and autonomous linear switching subsystems. We then prove that *UAS* of the controlled subsystem and uniform asymptotic stability of the autonomous subsystem imply *UAS* of the given system. Dual results on detectability-based state-space reductions have been recently established in [6].

The paper is organized as follows. In Section 2, we review definitions of switching systems and stabilizability. In Section 3, we introduce the notion of Hybrid Lyapunov function and state our main stabilizability conditions. In Section 4, we present structural reductions of the hybrid state space based on the notion of hybrid invariant subspace. In Section 5, we define a state space transformation that decouples the given system into controlled and autonomous linear switching subsystems. We show that the asymptotic stabilizability of the controlled subsystem and the asymptotic stability of the autonomous subsystem imply the asymptotic stabilizability of the given system. The minimality of this decomposition is proved in Section 6, where the concept of input to state equivalence for switching systems is defined. Section 7 offers concluding remarks.

2. Switching systems

We define the class of *linear switching systems* [5] using the general model of hybrid automata (see e.g. [13,19]).

The hybrid state ξ of a linear switching system is composed of two components: the discrete state q_i, belonging to a finite set Q, and the continuous state x, belonging to a linear space $\mathbb{R}^{n_i}$, whose dimension n_i depends on q_i. The evolution of the discrete state is governed by a Finite State Machine (FSM); a transition $e = (q_i, \sigma, q_h)$ may occur at time t from the discrete state q_i to the discrete state q_h, if the discrete disturbance σ occurs at time t. The evolution of the continuous state is described by a set of linear dynamic control systems, each associated with a discrete state. Whenever a transition e occurs, the continuous state x is instantly reset to a new value. More formally,

Definition 1. A linear switching system $\mathcal{S}$ is a tuple

$$(\Xi, \Theta, S, E, R),$$

where:

- $\Xi = \bigcup_{i\in J} \{q_i\} \times \mathbb{R}^{n_i}$ is the hybrid state space, where:
    - ∘ $Q = \{q_i, i \in J\}$ is the set of discrete states, $J = \{1, 2, \ldots, N\}$;
    - ∘ $\mathbb{R}^{n_i}$, $n_i \geq 0$, is the continuous state space associated with the discrete state $q_i \in Q$;
- $\Theta = \Sigma \times \mathbb{R}^m$ is the hybrid input space, where:
    - ∘ $\Sigma = \{\sigma_h, h \in J_1\}$ is the discrete disturbance space, $J_1 = \{1, 2, \ldots, N_1\}$;
    - ∘ $\mathbb{R}^m$, $m \geq 0$, is the continuous input space;

- S is a map associating with each discrete state $q_i \in Q$ the following linear dynamic control system:
$$\dot{x}(t) = f_i(x(t), u(t)),$$
where $x(t) \in \mathbb{R}^{n_i}$ is the state, $u(t) \in \mathbb{R}^m$ is the input at time $t \geq 0$ and the function f_i is linear on $\mathbb{R}^{n_i} \times \mathbb{R}^m$;
- $E \subset Q \times \Sigma \times Q$ is a collection of transitions;
- $R = \{R_e, e \in E\}$; for each $e = (q_i, \sigma, q_h) \in E$, $R_e : \mathbb{R}^{n_i} \to \mathbb{R}^{n_h}$ is the reset function, linear on $\mathbb{R}^{n_i}$.

A linear switching system $\mathcal{S}$ is said to be *autonomous* if each system $S(q_i)$ is an autonomous system, i.e. described by the equation $\dot{x}(t) = \overline{f}_i\,(x(t))$, with $\overline{f}_i$ linear on $\mathbb{R}^{n_i}$.

If $mn_i > 0$, then the system $S(q_i)$ can be equivalently rewritten as

$$\dot{x}(t) = A_i x(t) + B_i u(t),$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times m}$.

If $n_i > 0$ and $m = 0$, then $S(q_i)$ is an autonomous system described by the equation

$$\dot{x}(t) = A_i x(t).$$

Given $e = (q_i, \sigma, q_h) \in E$, if $n_i n_h > 0$ then there exists a matrix $M_e \in \mathbb{R}^{n_h \times n_i}$ such that $R_e\,(x) = M_e x$.

We now formally define the semantics of linear switching systems. First of all we assume throughout the paper that *the discrete disturbance is not available for measurements*, and that the class of admissible continuous inputs is the set $\mathcal{U}$ of piecewise continuous control functions $u : \mathbb{R} \to \mathbb{R}^m$. As defined in [13], a *hybrid time basis* τ is an infinite or finite sequence of sets $I_j = \{t \in \mathbb{R} : t_j \leq t \leq t'_j\}$, with $t'_j = t_{j+1}$; set card$(\tau) = L + 1$. If $L < \infty$, then t'_L can be finite or infinite. A hybrid time basis τ is said to be *finite*, if $L < \infty$ and $t'_L < \infty$ and *infinite*, otherwise. Given a hybrid time basis τ, any time instant t'_j is called *switching time*. Since linear switching systems are time invariant, we assume without loss of generality that $t_0 = 0$.

Throughout the paper, we assume that there is minimum time separation between two consecutive switching times:

Assumption 1 (*Minimum Dwell Time*). There exists a real $\delta_m > 0$, called minimum dwell time [15], such that for any hybrid time basis τ, $t'_j - t_j \geq \delta_m$, $j = 0, \ldots, L$.

Such assumption plays a key role in some of the results that we establish in this paper (in particular in Theorem 14 and in all results depending on it). However, for simplicity we will suppose that Assumption 1 holds throughout all the paper.

The existence of a minimum dwell time is a commonly used assumption in the analysis of switching systems (e.g. [15,12,7] and the references therein), and models the inertia of the system to react to an external (discrete) input. Denote by $\mathcal{T}$ the set of all hybrid time bases satisfying Assumption 1. The temporal evolution of a linear switching system can be now defined as follows:

Definition 2 (*Switching System Execution*). An execution χ of a linear switching system $\mathcal{S}$ is a collection

$$(\xi_0, \tau, \sigma, u, \xi),$$

with hybrid initial state $\xi_0 \in \Xi$, hybrid time basis $\tau \in \mathcal{T}$, discrete disturbance $\sigma : \mathbb{N} \to \Sigma$, control input $u \in \mathcal{U}$ and hybrid state evolution $\xi : \mathbb{R} \times \mathbb{N} \to \Xi$. The hybrid state evolution ξ is defined as follows:

$$\begin{aligned}
&\xi(0,0) = \xi_0,\\
&\xi(t,j) = (q\,(j)\,, x(t,j))\,, \quad t \in I_j, j = 0, 1, \ldots, L,\\
&\xi(t_{j+1}, j+1) = (q\,(j+1)\,, R_{e_j}(x(t'_j, j))), \quad j = 0, 1, \ldots, L-1,
\end{aligned}$$

where $q : \mathbb{N} \to Q$, and for any $j = 0, 1, \ldots, L-1$, $e_j = (q(j), \sigma(j), q(j+1)) \in E$ and $x(t, j)$ is the (unique) solution at time t of the dynamic system $S(q(j))$, with initial time t_j, initial condition $x(t_j, j)$ and continuous input u.

Remark 3. The class of linear switching systems is related to the class of *linear switched systems*, which has been extensively studied in the literature (see e.g. [18] and the references therein). While in a switching system transitions are caused by discrete disturbances, in a switched system they are caused by discrete inputs (i.e. discrete controls). A formal definition of switched systems can be obtained from Definition 1 by assuming that Σ is the set of discrete inputs. The semantics of switched systems is formally specified by Definition 2, where $\sigma : \mathbb{N} \to \Sigma$ is a discrete input function. The notion of switched systems obtained by Definition 1 generalizes the models of [18], where transitions are defined between every pair of discrete states and the reset matrix is the identity.

We abuse notation by using the same symbol x both for the state of $S(q)$ and for the continuous component of the hybrid state of $\mathcal{S}$. The context and the different arguments of the functions make the meaning of such a symbol univocally determined.

We focus on feedback control laws, where the control input u at time t depends on the values of the hybrid state at times $t' \leq t$. Given a linear switching system $\mathcal{S}$ and an execution χ, let the function $\eta : \mathbb{R} \to \Xi$ be defined as:

$$\eta(t) = \xi(t, j), \quad t \in [t_j, t'_j), \; j = 0, 1, \ldots, L.$$

We assume that *the hybrid state evolution is available for control synthesis*, and hence the set

$$\mathcal{Y} = \{\eta|_{[0,t]}, t \geq 0\},$$

embeds all the information on the hybrid state evolution available for control purposes.

A *control strategy* φ is a function $\varphi : \mathcal{Y} \to \mathbb{R}^m$ such that the function defined by $u(t) = \varphi(\eta|_{[0,t]}), t \geq 0$ belongs to $\mathcal{U}$. A switching system $\mathcal{S}$ together with a control strategy φ is called *controlled switching system* and its executions with $u(t) = \varphi(\eta|_{[0,t]}), t \geq 0$ are called *controlled executions.*

Given a switching system $\mathcal{S}$ and two sets

$$\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi,$$

$$\Omega' = \bigcup_{i \in J} \{q_i\} \times \Omega'_i \subset \Xi,$$

we define for any $\alpha \in \mathbb{R}$

$$\Omega + \Omega' := \bigcup_{i \in J} \{q_i\} \times \left(\Omega_i + \Omega'_i\right),$$

$$\alpha\Omega := \bigcup_{i \in J} \{q_i\} \times \alpha\Omega_i.$$

The set Ω inherits the properties of the sets $\Omega_i \subset \mathbb{R}^{n_i}$. In particular, Ω:

- is convex (resp. bounded), if Ω_i is convex (resp. bounded), $\forall i \in J$;
- has the origin as an interior point, if each set Ω_i has the origin as an interior point in $\mathbb{R}^{n_i}$.

Define the *hybrid unit ball* by:

$$\mathcal{B} = \bigcup_{i \in J} \{q_i\} \times \mathcal{B}_i,$$

where $\mathcal{B}_i = \{x \in \mathbb{R}^{n_i} : \|x\|_{n_i} \leq 1\}$ and $\|x\|_{n_i}$ denotes the Euclidean norm in $\mathbb{R}^{n_i}$.

3. Stabilizability, invariance and Lyapunov analysis

In this section, we introduce a notion of asymptotic stabilizability, uniform with respect to all possible switching sequences (hence called *UAS* in the introduction). Linearity of the dynamics implies that uniformity with respect to initial states in $\rho\mathcal{B}$ and globality can be assumed without loss of generality. Therefore, throughout the paper we call this property asymptotic stabilizability, for simplicity.

We extend some well-known results for stabilizability of linear systems to stabilizability of linear switching systems. In particular, we show the equivalence between asymptotic stabilizability and the existence of a suitable Lyapunov function and then we show that asymptotic stabilizability is equivalent to input-to-state stability, as defined in [17].

Definition 4 (*Asymptotic Stabilizability*). A linear switching system $\mathcal{S}$ is asymptotically stabilizable if there exists a control strategy φ (called stabilizing control strategy) such that the following conditions are satisfied:

(i) $\forall \varepsilon > 0, \exists \rho > 0$ such that $\xi(t, j) \in \varepsilon \mathcal{B}, \forall t \geq 0, \forall j = 0, 1, \ldots$ and for all controlled executions of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \rho \mathcal{B}$;

(ii) $\forall \varepsilon > 0, \forall \rho > 0, \exists \hat{t} > 0$ such that $\xi(t, j) \in \varepsilon \mathcal{B}, \forall t \in I_j \cap [\hat{t}, \infty), \forall j = \hat{j}, \ldots, L, \hat{j} = \min\{j : \hat{t} \in I_j\}$, for all controlled executions of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \rho \mathcal{B}$.

Since we are addressing stabilizability (an asymptotic property), in this definition we considered infinite executions, i.e. executions such that

$$\sum_{j=0}^{L} \left(t'_j - t_j \right) = \infty.$$

A switching system satisfying condition (i) in Definition 4 is said to be *stabilizable*. An asymptotically stabilizable autonomous linear switching system is said to be *asymptotically stable*. If condition (ii) holds with $\varepsilon = 0$, then $\mathcal{S}$ is called *controllable*. Furthermore given $\widetilde{J} \subset J$, if there exists a control strategy such that conditions (i) and (ii) of Definition 4 hold for all controlled executions of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \bigcup_{i \in \widetilde{J}} \{q_i\} \times \rho \mathcal{B}_i$, we say that $\mathcal{S}$ is *asymptotically stabilizable starting from* $\widetilde{Q} = \{q_i \in Q, i \in \widetilde{J}\}$.

Remark 5. By applying the definition above, the following result is established: a linear switching system $\mathcal{S}$ with minimum dwell time $\delta_m > 0$ is controllable if and only if any linear system $S(q_i)$, $q_i \in Q$ is controllable.

The definitions below generalize the concepts of invariant set, controlled invariant set, and domain of attraction (see [1]) to hybrid systems. Let **0** denote the identically zero function in the appropriate space.

Definition 6. Given a switching system $\mathcal{S}$, a set

$$\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi$$

is said to be:

- *$\mathcal{S}$-invariant* if, for any hybrid initial state $\xi_0 \in \Omega$ and for any execution of $\mathcal{S}$ with $u = \mathbf{0}$,
$$\xi(t, j) \in \Omega, \quad \forall t \in I_j, \ \forall j = 0, 1, \ldots, L; \tag{1}$$
- *controlled invariant,* if there exists a control strategy φ such that for any controlled execution of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \Omega$, condition (1) holds. We say that φ makes the set Ω invariant for system $\mathcal{S}$;
- a *β-domain of attraction* for some $\beta > 0$, if there exists a control strategy φ such that for any controlled execution of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \Omega$,
$$\xi\,(t, j) \in \mathrm{e}^{-\beta t}\Omega, \quad \forall t \in I_j, \forall j = 0, 1, \ldots, L.$$

Remark 7. The existence of a controlled invariant set (resp. domain of attraction) is deeply related to the stabilizability (resp. asymptotic stabilizability) property of the system. In particular, the existence of a controlled invariant set (resp. domain of attraction) for $\mathcal{S}$, that is bounded with the origin as an interior point, implies stabilizability (resp. asymptotic stabilizability) of the system $\mathcal{S}$. We will show that the converse implications hold as well.

Given some set $\Omega \subset \mathbb{R}^{n_i}$, $\mathcal{I}_i\,(\Omega)$ and $\mathcal{I}_{i\beta}\,(\Omega)$ denote the maximal controlled invariant subset of Ω and the maximal β-domain of attraction in Ω for system $S(q_i)$, respectively. Such maximal sets are well defined, since the properties are closed with respect to union of sets. Obviously $\mathcal{I}_{i0}\,(\Omega) = \mathcal{I}_i\,(\Omega)$.

The following results fully characterize $\mathcal{S}$-invariance, controlled invariant sets and domains of attraction for switching systems. Such characterization will be useful for assessing stabilizability of a switching system in terms of the existence of controlled invariant sets or domains of attraction. Given $e = (q_i, \sigma, q_h) \in E$, the symbol $R_e^{-1}\,(\Omega_h)$ denotes the inverse image of the set Ω_h through the function R_e. Let $x_i(t, x_0, u|_{[o,t)})$ be the controlled evolution at time t of the system $S(q_i)$, starting at $t = 0$ from the initial state x_0, with control u. The proof of the following first lemma is a straightforward generalization of results established in [5] and is therefore omitted. Consider a set:

$$\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi.$$

Lemma 8. *The set Ω is $\mathcal{S}$-invariant if and only if for any $x_0 \in \Omega_i$ the following conditions hold:*

(i) $x_i(t, x_0, \mathbf{0}) \in \Omega_i, \forall t \geq 0$ *and*
(ii) $R_e(x_i(t, x_0, \mathbf{0})) \in \Omega_h, \forall e = (q_i, \sigma, q_h) \in E, \forall t \geq \delta_m$.

Lemma 9. *The set Ω is a β-domain of attraction (resp. controlled invariant set) for the system $\mathcal{S}$ if and only if there exists a control strategy φ such that, by setting $u(t) = \varphi(\eta|_{[0,t]})$, $t \geq 0$, for any $i \in J$ and for any $x_0 \in \Omega_i$ the following conditions hold (resp. the following conditions hold with $\beta = 0$):*

(i) $x_i(t, x_0, u|_{[0,t)}) \in \mathrm{e}^{-\beta t}\Omega_i, \forall t \geq 0$ *and*
(ii) $x_i(t, x_0, u|_{[0,t)}) \in \mathrm{e}^{-\beta t}\mathcal{I}_{i\beta}\left(\bigcap_{j \in J_i}\left(R^{-1}_{(q_i,\sigma,q_j)}(\Omega_j)\right) \bigcap \Omega_i\right), \forall t \geq \delta_m,$

where $J_i = \{h \in J : (q_i, \sigma, q_h) \in E\}$.

Proof. For the characterization of controlled invariance, see [5]. As for the property of being a β-domain of attraction, given $\mathcal{S} = (\Xi, \Theta, S, E, R)$ set $\mathcal{S}' = (\Xi, \Theta, S', E, R)$, where S' associates to any $q \in Q$ the linear dynamical control system

$$S'(q) : \begin{cases} \dot{x}(t) = (A_i + \beta I)x(t) + B_i u(t), & \text{if } n_i > 0, \\ S(q), & \text{if } n_i = 0. \end{cases}$$

It is easily seen that Ω is a β-domain of attraction for the system $\mathcal{S}$ if and only if Ω is a controlled invariant for the system $\mathcal{S}'$. Therefore by linearity of $\mathcal{S}$ and $\mathcal{S}'$ the result follows. ■

The following theorem establishes the equivalence between stabilizability and the existence of a controlled invariant set and between asymptotic stabilizability and the existence of a domain of attraction.

Theorem 10. *A system $\mathcal{S}$ is stabilizable (resp. asymptotically stabilizable) if and only if there exists a bounded set with the origin as interior point, which is controlled invariant (resp. a β-domain of attraction for some $\beta > 0$) for $\mathcal{S}$.*

Proof. Equivalence between stabilizability of $\mathcal{S}$ and the existence of a bounded set with the origin as interior point, controlled invariant for $\mathcal{S}$ is a straightforward consequence of Definitions 4 and 6. As pointed out in Remark 7, the existence of a β-domain of attraction for $\mathcal{S}$, that is bounded with the origin as interior point, implies the asymptotic stabilizability of $\mathcal{S}$. Conversely, if $\mathcal{S}$ is asymptotically stabilizable, then there exists a sufficiently small $\beta > 0$ such that the system $\mathcal{S}'$, defined in the proof of Lemma 9, is stabilizable. Hence by the first part of this proof, there exists a bounded set with the origin as interior point, controlled invariant for $\mathcal{S}'$, which, as already pointed out in the proof of Lemma 9, is a β-domain of attraction for $\mathcal{S}$. ■

Given a switching system $\mathcal{S}$, a function $V : \Xi \to \mathbb{R}$ is continuous if $V((q_i, .))$ is continuous, for each $i \in J$.

The function V is said to be non-increasing (resp. strictly decreasing) for an execution χ of $\mathcal{S}$ if $V(\xi(., j))$ is non-increasing (resp. strictly decreasing) in I_j, $\forall j = 0, 1, \ldots, L$ and $V(\xi\left(t_{j+1}, j+1\right)) \leq V(\xi(t'_j, j))$, $\forall j = 0, 1, \ldots, L-1$.

A function $V : \Xi \to \mathbb{R}$ is said to be a *hybrid Lyapunov function* if it satisfies the following conditions:

- V is continuous;
- $\forall i \in J$, $V((q_i, .))$ has a unique minimum at the origin of $\mathbb{R}^{n_i}$.

Consider a bounded controlled invariant set (resp. a β-domain of attraction)

$$\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi,$$

with the origin as an interior point. Since the switching system is linear, the set Ω can be assumed w.l.o.g. to be convex.

Given Ω, consider the hybrid generalization of the Minkowski functional $\Psi_\Omega : \Xi \to \mathbb{R}$ (see [10]):

$$\Psi_\Omega(\xi) = \inf\{\mu \in \mathbb{R}, \mu \geq 0 : \xi \in \mu\Omega\}. \tag{2}$$

By the above definition, $\Psi_{\Omega}(\xi) < 1$ (resp. $\Psi_{\Omega}(\xi) = 1$, $\Psi_{\Omega}(\xi) > 1$), if and only if $\xi \in int(\Omega)$ (resp. $\xi \in \partial\Omega$, $\xi \notin \Omega$), where:

$$\operatorname{int}(\Omega) = \bigcup_{i \in J} \{q_i\} \times \operatorname{int}(\Omega_i) \subset \Xi,$$

$$\partial\Omega = \bigcup_{i \in J} \{q_i\} \times \partial\Omega_i \subset \Xi.$$

By (2), $\Psi_{\Omega}((q_i, .))$ has a unique minimum at the origin of $\mathbb{R}^{n_i}$; furthermore by convexity of Ω, function Ψ_{Ω} is continuous and therefore Ψ_{Ω} is a hybrid Lyapunov function. The use of the Minkowski functional allows defining 'set induced'-Lyapunov Functions, i.e. Lyapunov functions induced by controlled invariant sets or by domains of attraction. In fact this approach has been inspired by [1], where non-quadratic Lyapunov functions were set and characterized for the class of linear dynamic control systems.

The above discussion proves the necessity of the following result:

Theorem 11. *A switching system $\mathcal{S}$ is stabilizable (resp. asymptotically stabilizable) if and only if there exist a control strategy and a hybrid Lyapunov function that is not increasing (resp. strictly decreasing) for any controlled execution of $\mathcal{S}$.*

Proof. (Necessity) Since Ω is controlled invariant (resp. β-domain of attraction, for some $\beta > 0$) for $\mathcal{S}$, there exists a control strategy such that the function $\Psi_{\Omega}(.)$ is not increasing (resp. strictly decreasing) for any controlled execution of $\mathcal{S}$.

(Sufficiency) Let V be a hybrid Lyapunov function. Since each function $V((q_i, .))$ is continuous and attains its unique minimum at the origin of $\mathbb{R}^{n_i}$, then $\forall r > 0$ the set $\Omega^r = \{\xi \in \Xi : V(\xi) \leq r\}$ has the origin in the interior, and there exists a value $R > 0$ such that $\forall r_1, r_2$, $0 < r_1 < r_2 \leq R$, $\Omega^{r_1} \subset \Omega^{r_2}$ and both sets are compact. The function V is continuous and therefore $\forall r_1, r_2$, $0 < r_1 < r_2 \leq R$, $\exists \lambda \in (0, 1)$ such that $\Omega^{r_1} \subset \lambda\Omega^{r_2}$. In the following we show that stabilizability and asymptotic stabilizability hold locally, i.e. in a compact subset of the hybrid state space Ξ; the linearity in the dynamics and in the reset makes the results also true globally, i.e. in the whole hybrid state space Ξ.

(Stabilizability) Consider an execution of the switching system. Let us assume $\xi(0, 0) \in \Omega^r$, $r \in (0, R]$. Since there exists a control strategy such that V is not increasing for any controlled execution of $\mathcal{S}$, then $\xi(t, j) \in \Omega^r$, $\forall t \in I_j, \forall j = 0, \ldots, L$ and hence $\mathcal{S}$ is stabilizable.

(Asymptotic stabilizability) There exists a control strategy such that V is strictly decreasing for any controlled execution of $\mathcal{S}$. Consider a controlled execution of the switching system. Let $t_0 = 0$. Given some $j = 0, \ldots, L - 1$ and some $\widehat{t} \geq t_j + \delta_m$, let us assume $\xi(\widehat{t}, j) \in \Omega^r$, $r \in (0, R]$. Given $d, T \in \mathbb{R}$, $0 < d < T < 2\delta_m$, since Ω^r is compact and since at most a switching is allowed in $[\widehat{t}, \widehat{t} + T]$, then there exists $\lambda \in (0, 1)$, not depending on $\xi(\widehat{t}, j)$, such that $\xi(\widehat{t} + T, h) \in \lambda\Omega^r$, being $h = j$, if no switchings occurred in $[\widehat{t}, \widehat{t} + T]$ or $h = j + 1$, otherwise. Therefore there exists a control strategy such that the hybrid state converges asymptotically to the set $\bigcup_{i \in J} \{q_i\} \times \{0\} \subset \Xi$, $\forall \xi(\widehat{t}, j) \in \Omega^r$. Thus $\mathcal{S}$ is asymptotically stabilizable. ■

Remark 12. In the particular case of switching systems with identity reset, a Hybrid Lyapunov Function satisfying Theorem 11 corresponds to a Multiple Lyapunov Function, as defined in [3]. Conversely, a Multiple Lyapunov Function is not in general a Hybrid Lyapunov Function satisfying Theorem 11, since in Theorem 11 we require that the function is decreasing along any controlled trajectory. Hence, the concept of Multiple Lyapunov Function is weaker. However, in Theorem 11 we state the *necessity* of a Hybrid Lyapunov Function for the stability of the controlled switching system, while the existence of a Multiple Lyapunov Function is sufficient for the stability of the controlled switching system [3]. An advantage in defining a function from the hybrid state space to the set of reals, as we do here with the Hybrid Lyapunov function, is that we obtain a more compact formulation of the results; our definition of Hybrid Lyapunov Function and Theorem 11 are stated in a very simple way, if compared to the analogous sufficient results in [3] (see Definition 2.2 and Theorem 2.3).

Remark 13. As a consequence of Lemma 8, the fact that the reset function is the identity *does not* imply the existence of a set Ω, not depending on q_i, such that $\bigcup_{i \in J} \{q_i\} \times \Omega$ is a β-domain of attraction (resp. controlled invariant set). This implication holds (hence there exists a *common Lyapunov function*) if some particular conditions hold: no condition is given on the dwell time, non-Zeno behaviour is assumed, the reset is the identity function and discrete transitions are arbitrary (i.e. no discrete structure is considered, so that a transition is defined between any two discrete states). Under these particular conditions, Theorem 11 becomes equivalent to converse Theorems of [4]. Under the same conditions, our stabilizability problem becomes a special case of the one addressed in [2]: more precisely, for the class of polytopic linear parameter varying (LPV) systems, in this last paper the existence of a stabilizing controller was proved to be equivalent to the existence of a robust stabilizing controller, i.e. depending on the continuous state, but not on the discrete state. Therefore, for LPV systems, there exists a hybrid Lyapunov function if and only if there exists a *robust common Lyapunov function*. Moreover, we assume a positive minimum dwell time in our paper. Therefore the class of systems we are considering does not fall in the class of LPV systems, nor in the class of the systems addressed in [4].

We conclude this section by deriving a property of the 'bounded input-bounded state'-type for linear switching systems subject to continuous disturbances. More precisely, we consider a system $\mathcal{S}^d = \left(\Xi, \Theta, S^d, E, R^d\right)$ as in Definition 1, where S^d is a map associating to any discrete state $q_i \in Q$ the following system:

$$\dot{x}(t) = g_i\left(x(t), u(t), w(t)\right),$$

where $w \in \mathcal{D}$ is the disturbance function, $\mathcal{D}$ is the set of piecewise continuous functions $w : \mathbb{R} \to \mathbb{R}^p$, and g_i is linear in its arguments. We suppose that a disturbance affects also the reset, i.e. for any execution of $\mathcal{S}^d$, $\xi\left(t_{j+1}, j+1\right) = \left(q\left(j+1\right), R^d_{e_j}(x(t'_j, j), w(t'_j))\right)$, with the reset function $R^d_{e_j}$ being linear in its arguments. System $\mathcal{S}^d$ will be said to be asymptotically stabilizable if it is asymptotically stabilizable for $w(t) = 0$, $\forall t \geq 0$.

The first statement of the next theorem says that asymptotic stabilizability of the switching system $\mathcal{S}^d$ implies *ISS* of the controlled switching system, for any stabilizing control strategy. Note that in general the controlled switching system is nonlinear and time varying, since we are assuming a non-zero minimum dwell time. In the second statement, the case of a vanishing disturbance is addressed.

Theorem 14. *Let $\mathcal{S}^d$ be asymptotically stabilizable. Then there exists a control strategy such that*

$$\forall r > 0, \qquad \forall \mu_0 > 0, \qquad \exists \gamma \geq 1 : \xi(t, j) \in \gamma \mathcal{B}, \qquad \forall t \in I_j, \qquad \forall j = 0, 1, \ldots, L,$$

for all disturbance functions $w \in \mathcal{D}$, with $\|w(t)\| \leq r$, $\forall t \geq 0$, for all controlled executions of $\mathcal{S}^d$ with hybrid initial state in $\mu_0 \mathcal{B}$.
If moreover $\|w(t)\| \leq e^{-\lambda t} r$, $\forall t \geq 0$, for some $\lambda > 0$ then there exists a control strategy and $\widehat{\lambda} \in (0, \lambda)$ such that

$$\forall r > 0, \qquad \forall \mu_0 > 0, \qquad \exists \gamma \geq 1 : \xi(t, j) \in \gamma e^{-\widehat{\lambda} t} \mathcal{B}, \qquad \forall t \in I_j, \qquad \forall j = 0, 1, \ldots, L,$$

for all controlled executions of $\mathcal{S}^d$ with hybrid initial state in $\mu_0 \mathcal{B}$.

The proof is quite technical and does not offer a particular insight that can be used in the rest of the paper. Hence, it is reported in the Appendix.

4. Invariant hybrid subspaces

In this section, we introduce an invariant linear hybrid subspace that will be the basis of the asymptotic stabilizability analysis for linear switching systems.

A set $\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi$ is a hybrid linear subspace of Ξ if Ω_i is a linear subspace of $\mathbb{R}^{n_i}$, for any $i \in J$. In this paper, for the sake of simplicity, we refer to a hybrid linear subspace as a subspace. The following result gives a necessary and sufficient condition for a subspace to be $\mathcal{S}$-invariant.

Proposition 15. *Given a switching system $\mathcal{S}$, a subspace $\bigcup_{i \in J} \{q_i\} \times \Omega_i$ is $\mathcal{S}$-invariant if and only $\forall x \in \Omega_i$, $\forall i \in J$ the following conditions hold:*

- *$f_i\left(x, 0\right) \in \Omega_i$;*
- *$R_e(x) \in \Omega_h$, $\forall x \in \Omega_i$, $\forall e = (q_i, \sigma, q_h) \in E$.*

The proof of this proposition is a straightforward consequence of Lemma 8 and is therefore omitted. Since the intersection of any two $\mathcal{S}$-invariant subspaces is an $\mathcal{S}$-invariant subspace, the minimal $\mathcal{S}$-invariant subspace containing a given subspace is well defined.

Let

$$\mathcal{G} = \bigcup_{i \in J} \{q_i\} \times \mathcal{G}_i \tag{3}$$

be the minimal $\mathcal{S}$-invariant subspace containing

$$\mathcal{H} = \bigcup_{i \in J} \{q_i\} \times H_i, \tag{4}$$

where:

$$H_i = \begin{cases} \operatorname{Im}(B_i), & \text{if } mn_i > 0 \\ \mathbb{R}^0, & \text{otherwise.} \end{cases}$$

The following result illustrates a procedure for computing $\mathcal{G}$ in a finite number of steps.

Theorem 16. *Given $\mathcal{S}$, define the sequence of subspaces $\Omega_i^k \subset \mathbb{R}^{n_i}$, $k = 0, 1, 2, \ldots, i \in J$, as*

$$\Omega_i^0 = \operatorname{Im}(B_i \ A_i B_i \ \ldots \ A_i^{n_i - 1} B_i),$$

$$\Omega_i^k = \sum_{h=0}^{n_i - 1} (A_i)^h \, \Phi_i^k$$

$$\Phi_i^k = \sum_{j \in J_i^-} \left\{ R_{(q_j, \sigma, q_i)}(x), x \in \Omega_j^{k-1} \right\} + \Omega_i^{k-1},$$

where $J_i^- = \left\{ j \in J : \left(q_j, \sigma, q_i\right) \in E \right\}$. The sequence $\{\Omega_i^k, i \in J\}_{k=0,1,2,\ldots}$ converges in $k^* \leq \sum_{i=1}^N n_i$ steps and

$$\mathcal{G} = \bigcup_{i \in J} \{q_i\} \times \Omega_i^{k^*}.$$

Proof. By definition of $\mathcal{G}$ and recalling that by Caley–Hamilton Theorem Ω_i^k is the minimal subspace of $\mathbb{R}^{n_i}$ containing Φ_i^k, it follows that $\Omega_i^k \subset \mathcal{G}_i$, $\forall k \in \mathbb{N}, \forall i \in \mathbb{N}$. By finite dimensionality of the linear subspaces involved, it is clear that the sequence $\{\Omega_i^k, i \in J\}_{k=0,1,2,\ldots}$ converges in $n \leq \sum_{i=1}^N n_i$ steps to some collection of subspaces $\{F_i, i \in J\}$, and by Proposition 15, the set $\bigcup_{i \in J} \{q_i\} \times F_i$ is $\mathcal{S}$-invariant. Moreover, by construction, $\mathcal{H} \subset \bigcup_{i \in J} \{q_i\} \times F_i$. Since $\mathcal{G}$ is the minimal $\mathcal{S}$-invariant containing the set $\mathcal{H}$, then the result follows. ■

Remark 17. The notion of maximal controlled invariant set of switched linear systems was introduced in [9], while here we introduce the notion of minimal $\mathcal{S}$-invariant subspace. The concept of invariance is the same in both cases. However, no relationship exists between the sets that are obtained in the two cases because in our case switching is uncontrollable while in [9], switching control is used to render the sets invariant.

Next we provide a link between results presented in this section and results established in [18], in the framework of switched systems.

By definition, the discrete evolution of the switching system $\mathcal{S} = (\Xi, \Theta, S, E, R)$ is described by the FSM (Q, Σ, E). We recall that the FSM is said to be *strongly connected* if there exists a path between any pair of discrete states in Q. The following result holds as a direct consequence of Proposition 15 and Theorem 16.

Corollary 18. *Consider a switching system $\mathcal{S} = (\Xi, \Theta, S, E, R)$ and suppose that:*

(i) *$n_i = n$, $\forall i \in J$;*
(ii) *$R_e(x) = x$, $\forall e \in E$, $\forall x \in \mathbb{R}^n$;*
(iii) *(Q, Σ, E) is strongly connected.*

Then

$$\mathcal{G} = Q \times \widehat{\mathcal{G}},$$

where $\widehat{\mathcal{G}} \subset \mathbb{R}^n$ *is the minimal linear subspace of* $\mathbb{R}^n$ *satisfying for any* $i \in J$ *the following conditions:*

$$A_i\widehat{\mathcal{G}} \subset \widehat{\mathcal{G}}; \qquad \mathrm{Im}(B_i) \subset \widehat{\mathcal{G}}.$$

Remark 19. The subspace $\widehat{\mathcal{G}}$ coincides with the 'multiple controllable subspace', as defined in [18] in the framework of switched linear systems (see also Remark 3).

5. State space reductions based on stabilizability

It is well-known that a linear system S is asymptotically stabilizable if and only if a suitable subsystem extracted from S is asymptotically stable. For general switching systems, asymptotic stabilizability conditions become more involved. In this section, we show how to extract from a given linear switching system $\mathcal{S}$ two subsystems so that the asymptotic stabilizability of one of them and the asymptotic stability of the other one imply the asymptotic stabilizability of $\mathcal{S}$.

Our procedure is based on the reduction of the state space of the linear switching system $\mathcal{S}$ by means of the invariant hybrid subspace $\mathcal{G}$, as defined in the previous section.

We first focus on the discrete state space decomposition problem.

Given a linear switching system $\mathcal{S} = (\Xi, \Theta, S, E, R)$, define the restriction of $\mathcal{S}$ to a subset Q' of Q as a linear switching system:

$$\mathcal{S}|_{Q'} = \left(\Xi', \Theta, S', E', R'\right), \tag{5}$$

where:

$$\begin{aligned} &\Xi' = \bigcup_{q_i \in Q'} \{q_i\} \times \mathbb{R}^{n_i};\\ &S'(q) = S(q), \quad \forall q \in Q';\\ &E' = \{(q_i, \sigma, q_h) \in E : q_i, q_h \in Q'\};\\ &R'_e = R_e, \quad \forall e \in E'. \end{aligned}$$

The next proposition is a specialization of results established in [8]. It shows a method for reducing the discrete state space of a given switching system, while preserving asymptotic stabilizability of the systems involved.

Proposition 20 (*[8]*). *Let* $\mathcal{S}$ *be asymptotically stabilizable starting from* $\widetilde{Q}$. *Then* $\mathcal{S}$ *is asymptotically stabilizable if and only if the linear switching system* $\mathcal{S}|_{\widehat{Q}}$, *where* $\widehat{Q} = \{q_i \in Q : q_i \notin \widetilde{Q}\}$, *is asymptotically stabilizable.*

The proposition above implies that $\mathcal{S}$ is asymptotically stabilizable if and only if the linear switching system, obtained from $\mathcal{S}$ by skipping all discrete states associated with

- controllable linear systems, or
- with zero-dimensional linear systems,

is asymptotically stabilizable. Therefore we can assume w.l.o.g. that for any discrete state $q_i \in Q$, system $S(q_i)$ is not controllable and that $n_i > 0$.

Since we will be presenting a decomposition of the switching system into a switching system with controls and an autonomous switching system, we suppose w.l.o.g. that $\mathcal{S}$ is non-autonomous.

Given the hybrid invariant subspace $\mathcal{G}$ as in (3), let $0 \le \mu_i \le n_i$ be the dimension of $\mathcal{G}_i$ and define a hybrid state space transformation for $\mathcal{S}$, as follows. For each $i \in J$, consider the matrix:

$$T_i = \left(b_1^i \dots b_{\mu_i}^i \quad v_1^i \dots v_{n_i-\mu_i}^i\right) \in \mathbb{R}^{n_i \times n_i},$$

where the vectors $b_1^i, \dots, b_{\mu_i}^i$ are a basis for $\mathcal{G}_i$ and the vectors $v_1^i, \dots, v_{n_i-\mu_i}^i$ are such that T_i is full rank. Then the matrices

$$\begin{aligned} &\widehat{A}_i = T_i^{-1} A_i T_i,\\ &\widehat{B}_i = T_i^{-1} B_i,\\ &\widehat{M}_e = T_h^{-1} M_e T_i, \quad e = (q_i, \sigma, q_h) \end{aligned}$$

take the form:

$$\widehat{A}_i = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} \\ 0 & A_i^{(22)} \end{pmatrix}, \qquad \widehat{B}_i = \begin{pmatrix} B_i^{(1)} \\ 0 \end{pmatrix}, \qquad \widehat{M}_e = \begin{pmatrix} M_e^{(11)} & M_e^{(12)} \\ 0 & M_e^{(22)} \end{pmatrix}, \tag{6}$$

where, if $\mu_i > 0$, $A_i^{(11)} \in \mathbb{R}^{\mu_i \times \mu_i}$ and $B_i^{(1)} \in \mathbb{R}^{\mu_i \times m}$. If $\mu_i \mu_h > 0$ then $M_e^{(11)} \in \mathbb{R}^{\mu_h \times \mu_i}$.

Remark 21. In general, the pair $(A_i^{(11)}, B_i^{(1)})$ is not controllable.

The switching system obtained after the hybrid state-space transformation is algebraically equivalent [16] to the switching system $\mathcal{S}$ and therefore asymptotic stabilizability is preserved. Thus, there is no loss of generality if we assume that the dynamic matrices of the switching system $\mathcal{S}$ are of the form (6). We define the linear switching system $\mathcal{S}_c$ as:

$$\mathcal{S}_c = (\Xi^c, \Theta, S^c, E, R^c),$$

where $\Xi^c = \bigcup_{i \in J} \{q_i\} \times \mathbb{R}^{\mu_i}$, $S^c(q_i)$ is described by the dynamics

$$\dot{z}(t) = f_i^c(z(t), u(t)),$$

where, if $\mu_i > 0$, $f_i^c(z(t), u(t)) = A_i^{(11)} z(t) + B_i^{(1)} u(t)$ and, if $\mu_h \mu_i > 0$, $R_e^c(x) = M_e^{(11)} x$, $e = (q_i, \sigma, q_h)$.

We also define the autonomous linear switching system $\mathcal{S}_a$ as:

$$\mathcal{S}_a = (\Xi^a, \Theta, S^a, E, R^a),$$

where $\Xi^a = \bigcup_{i \in J} \{q_i\} \times \mathbb{R}^{n_i - \mu_i}$, $S^a(q_i)$ is described by the dynamics

$$\dot{z}(t) = f_i^a(z(t), 0),$$

where, if $n_i - \mu_i > 0$, $f_i^a(z(t)) = A_i^{(22)} z(t)$ and, if $(n_h - \mu_h)(n_i - \mu_i) > 0$, $R_e^a(x) = M_e^{(22)} x$, $e = (q_i, \sigma, q_h)$.

The following holds:

Theorem 22. *The switching system $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_c$ is asymptotically stabilizable and $\mathcal{S}_a$ is asymptotically stable.*

Proof. (Necessity) Since starting from $\mathcal{G}$, any execution of $\mathcal{S}$ remains in $\mathcal{G}$, then the system $\mathcal{S}_c$ has to be asymptotically stabilizable. The necessity for the asymptotic stability of $\mathcal{S}_a$ is obvious, by its own definition. (Sufficiency) We give the proof for the case $\mu_i > 0$, $\forall i \in J$. (the case $\mu_i = 0$, for some $i \in J$, can be solved with the same tools, but with more cumbersome notations). Consider the linear switching system $\mathcal{S}_{\text{coupled}} = (\Xi_c, \Theta, S_{\text{coupled}}, E, R_c)$, where for any $q_i \in Q$, $S_{\text{coupled}}(q_i)$ is described by the equations

$$\dot{z}(t) = A_i^{(11)} z(t) + B_i^{(1)} u(t) + A_i^{(12)} x_2(t, j),$$

with the hybrid state at $t \in I_j$ denoted by $\xi_1(t, j) = (q(j), x_1(t, j))$ and with the state after the transition e equal to

$$M_e^{(11)} x_1(t_j', j) + M_e^{(12)} x_2(t_j', j).$$

Since $\mathcal{S}_a$ is asymptotically stable, by Theorem 10 there exists a convex bounded set

$$\mathcal{W} = \bigcup_{i \in J} \{q_i\} \times \mathcal{W}_i \subset \Xi^a$$

with the origin as an interior point, that is a β-domain of attraction for $\mathcal{S}_a$, for some $\beta > 0$. Therefore for some suitable p there exists a set $W \subset \mathbb{R}^p$ and suitable matrices D_i and F_e such that

$$A_i^{(12)} \mathcal{W}_i \subset D_i W, \qquad M_e^{(12)} \mathcal{W}_i \subset F_e W.$$

Therefore by applying Theorem 14, and by linearity of the switching system, we conclude that the condition is sufficient. ■

This theorem establish a link to the classical *Kalman decomposition* of linear systems.

We consider now a special case. We first need to introduce the particular class of *static hybrid linear state feedback* control strategies φ, characterized by the condition that for any discrete state $q_i \in Q$, there exists a linear function $K_i : \mathbb{R}^m \to \mathbb{R}^{n_i}$ such that $\varphi(\eta|_{[0,t]}) = K_i\,(x(t, j))$. A switching system $\mathcal{S}$ is said to be *asymptotically stabilizable via static hybrid linear state feedback* if it is asymptotically stabilizable and the stabilizing control strategy is a static hybrid linear state feedback.

The next result shows that, under appropriate assumptions, the switching system $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_a$ is asymptotically stable.

For any $i \in J$, let

$$\mathcal{C}_i = (B_i \; A_i B_i \; \dots \; A_i^{n_i - 1} B_i),$$

be the controllability matrix associated with the linear system $S(q_i)$ and set

$$\mathcal{R} = \bigcup_{i \in J} \{q_i\} \times \mathcal{R}_i,$$

where $\mathcal{R}_i = \operatorname{Im}(\mathcal{C}_i)$. The following result holds.

Proposition 23. *Suppose that*

$$\mathcal{G} = \mathcal{R}, \tag{7}$$

then $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_a$ is asymptotically stable. Moreover, in this case, $\mathcal{S}$ is asymptotically stabilizable via static hybrid linear state feedback.

Proof. For sake of simplicity, assume $\mu_i > 0$. The necessary condition holds by Theorem 22. Condition (7) implies that the switching system $\mathcal{S}_c$ is controllable and then by [15], $\mathcal{S}_c$ is asymptotically stabilizable via static hybrid linear state feedback with appropriate gain matrices $K_1^1, K_2^1, \dots, K_N^1$. By applying Theorem 14 we conclude that $\mathcal{S}$ is asymptotically stabilizable and hence by setting $K_i = (K_i^1 \; 0_i)$, $i = 1, \dots, N$, where matrices 0_i are of appropriate dimensions, we obtain that $\mathcal{S}$ is asymptotically stabilizable via static hybrid linear state feedback with gain matrices $K_1, K_2, \dots, K_N$. ■

Remark 24. Writing the system equations as in (6) is possible also in the case of switched systems, but decoupling the system into controlled and autonomous subsystems is not possible, in general, since the transitions are controlled. In fact [18] addresses the special class of switched systems $\mathcal{S}$, where the reset function is the identity, and where the continuous dynamical systems share the same matrix A, i.e. $A_i = A, \forall i \in J$. Therefore $\mathcal{S}_a$ reduces to an autonomous linear system (hence its behaviour is independent from the transitions sequence) and it is shown that the asymptotic stability of $\mathcal{S}_a$ implies the stabilizability of the given switched system, but just in this special case.

6. Input to state equivalence

In this section, a relationship between the systems $\mathcal{S}_c$ and $\mathcal{S}$ is established. To do so, in the next definition we formalize the concept of input to state equivalence for two switching systems, starting from initial states with zero continuous component, as the existence of a bijection between the state trajectories of the systems.

Consider any two hybrid spaces:

$$\Xi = \bigcup_{i \in J} \{q_i\} \times \mathbb{R}^{n_i},$$

$$\Xi' = \bigcup_{h \in J'} \left\{q_h'\right\} \times \mathbb{R}^{n_h'}.$$

Let $\mathcal{F}$ be the class of functions $\psi : \Xi \to \Xi'$ which, for any $i \in J$, map $\{q_i\} \times \mathbb{R}^{n_i}$ onto $\left\{q_h'\right\} \times \mathbb{R}^{n_h'}$, i.e. which satisfy the following implication

$$\begin{gathered}\exists x \in \mathbb{R}^{n_i}, \exists x' \in \mathbb{R}^{n_h'}, \exists q_h' \in Q' : \psi\,(q_i, x) = \left(q_h', x'\right) \\ \Downarrow \\ \forall x \in \mathbb{R}^{n_i}, \exists x' \in \mathbb{R}^{n_h'} : \psi\,(q_i, x) = \left(q_h', x'\right),\end{gathered}$$

and are linear, in their second argument, i.e.

$$\forall \left(q_i, x\right), \left(q_i, z\right) \in \Xi,$$
$$\psi\left(q_i, \alpha x + \beta z\right) \Rightarrow \alpha\psi\left(q_i, x\right) + \beta\psi\left(q_i, z\right), \quad \forall \alpha, \beta \in \mathbb{R},$$

where, given $\xi = (q, x) \in \Xi$, the symbol $\alpha\xi$, $\alpha \in \mathbb{R}$, denotes $(q, \alpha x)$, and, given $\xi_1 = (q, x_1)$, $\xi_2 = (q, x_2) \in \Xi$, the symbol $\alpha_1\xi_1 + \alpha_2\xi_2$ denotes $(q, \alpha_1 x_1 + \alpha_2 x_2)$.

Definition 25. Two linear switching systems $\mathcal{S} = (\Xi, \Theta, S, E, R)$ and $\mathcal{S}' = \left(\Xi', \Theta', S', E', R'\right)$ are 0-state input to state equivalent (shortly IS-equivalent) if $\Theta' = \Theta$ and if there exist functions $\psi : \Xi \to \Xi'$ and $\psi' : \Xi' \to \Xi$, $\psi, \psi' \in \mathcal{F}$, such that:

(i) for any execution $\chi = (\xi_0, \tau, \sigma, u, \xi)$ of $\mathcal{S}$, with $\xi_0 \in \bigcup_{i \in J} \{q_i\} \times \{0\}$, there exists an execution $\chi' = \left(\xi_0', \tau', \sigma, u, \xi'\right)$ of $\mathcal{S}'$, with $\xi_0' \in \bigcup_{i \in J'} \left\{q_i'\right\} \times \{0\}$, such that $\eta'(t) = \psi(\eta(t))$, $\forall t \in [0, t_L)$, and, conversely,
(ii) for any execution $\chi' = \left(\xi_0', \tau', \sigma, u, \xi'\right)$ of $\mathcal{S}'$, with $\xi_0' \in \bigcup_{i \in J'} \left\{q_i'\right\} \times \{0\}$ there exists an execution $\chi = (\xi_0, \tau, \sigma, u, \xi)$ of $\mathcal{S}$, with $\xi_0 \in \bigcup_{i \in J} \{q_i\} \times \{0\}$, such that $\eta(t) = \psi'\left(\eta'(t)\right)$, $\forall t \in [0, t_L)$.

The definition above implies that if $\mathcal{S}$ and $\mathcal{S}'$ are IS-equivalent then $\mathrm{card}(Q) = \mathrm{card}(Q')$. It is possible to show that, in general, the cardinality of the set E is not preserved. We do not dwell more on this point, since we are going to prove the IS-equivalence of the systems $\mathcal{S}$ and $\mathcal{S}_c$, which share the same discrete state space Q and the same transitions set E. In fact,

Proposition 26. *The system $\mathcal{S}_c$ is IS-equivalent to $\mathcal{S}$ and has minimal dimensions, i.e. in any other system that is IS-equivalent to $\mathcal{S}$, the state dimension of each dynamic system is greater than or equal to the dimension of the corresponding dynamic system in $\mathcal{S}_c$.*

Proof. The IS-equivalence is straightforward, by definition of $\mathcal{S}_c$. The hybrid state of the system $\mathcal{S}$ has to evolve on a $\mathcal{S}$-invariant set Ω, for any input function. Hence Ω has to contain the hybrid subspace $\mathcal{H}$. By linearity of the systems, the minimal subspace containing Ω is itself $\mathcal{S}$-invariant. Therefore $\mathcal{S}_c$ is minimal, since $\mathcal{G}$ is the minimal $\mathcal{S}$-invariant subspace, containing the hybrid subspace $\mathcal{H}$. ■

7. Conclusion

In this paper, we showed that asymptotic stabilizability of linear switching systems is equivalent to the existence of a hybrid Lyapunov function for the controlled system, for a suitable control strategy. Further, we proved that asymptotic stabilizability of a switching system with minimum dwell time, is equivalent to *ISS* of the controlled switching system, with a stabilizing control law. We then derived structural reductions of the hybrid state space, which allow the decomposition of the original problem into simpler subproblems. Moreover, we derived the relationships between the well-known Kalman decomposition of linear dynamic control systems and our results.

Appendix. Proof of Theorem 14

We assume without loss of generality that $n_i > 0$, $\forall i \in J$ (the case $n_i = 0$, for some $i \in J$ can be solved with the same tools, but with more cumbersome notations). Therefore continuous dynamics associated to $\mathcal{S}^d$ can be described by the equation

$$\dot{x}(t) = A_i x(t) + B_i u(t) + D_i w(t),$$

with matrices of appropriate dimensions and

$$\xi\left(t_{j+1}, j+1\right) = \left(q\left(j+1\right), M_{e_j} x(t_j', j) + F_{e_j} w(t_j')\right),$$

with $F_{e_j} \in \mathbb{R}^{n_j \times p}$. Since $\mathcal{S}$ is asymptotically stabilizable, then by Theorem 10 there exists a convex, bounded set with the origin as an interior point $\Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i$, that is a β-domain of attraction for $\mathcal{S}$, for some $\beta > 0$. For the same reason, we can assume w.l.o.g. that the matrices A_i are Hurwitz. If $\lambda = 0$, set $\widehat{\lambda} = 0$. Otherwise set

$\widehat{\lambda} \in (0, \min\{\lambda, \beta\})$ be such that $\widehat{A}_i = A_i + \widehat{\lambda} I$ is Hurwitz, $\forall i \in J$. Let $W \subset \mathbb{R}^p$ be the set $\{w \in \mathbb{R}^p : \|w\| \le r\}$. Since $w(t) \in \mathrm{e}^{-\lambda t} W, \forall t \ge 0$, then $w(t) \in \mathrm{e}^{-\widehat{\lambda} t} W, \forall t \ge 0$. By setting $z(t) = \mathrm{e}^{\widehat{\lambda} t} x(t)$, the system:

$$\dot{x}(t) = A_i x(t) + D_i w(t), \qquad w(t) \in \mathrm{e}^{-\widehat{\lambda} t} W, \quad \forall t \ge 0, \tag{8}$$

can be rewritten as:

$$\dot{z}(t) = \widehat{A}_i z(t) + D_i v(t), \quad v(t) \in W, \ \forall t \ge 0.$$

Since $\widehat{A}_i$ is Hurwitz and W is bounded, then the set R_i, reachable by the disturbance v, is bounded. Let Λ_i be the convex extension of the set R_i. By construction $x(t) = \mathrm{e}^{-\widehat{\lambda} t} z(t), t \ge 0$ and therefore for any function w, with $w(t) \in \mathrm{e}^{-\widehat{\lambda} t} W, \forall t \ge 0$, it follows that $x(t)$, solution of Eq. (8), starting from zero initial state, satisfies the inclusion $x(t) \in \mathrm{e}^{-\widehat{\lambda} t} \Lambda_i, \forall t \ge 0$. Let $\Lambda = \bigcup_{i \in J} \{q_i\} \times \Lambda_i$ and $\widehat{\alpha} = \max_{e \in E} \alpha(e)$, where, given $e = (q_i, \sigma, q_j)$, $\alpha(e)$ is such that $F(e)d \in \alpha(e) \Omega_j, \forall d \in W$. Then there exist $\psi \in (0, 1)$ and $\rho > 0$ such that

$$\mathrm{e}^{-\beta t} \rho \Omega + \mathrm{e}^{-\widehat{\lambda} t} \Lambda \subset \psi \mathrm{e}^{-\widehat{\lambda} t} \rho \Omega, \quad \forall t \ge \delta_m, \tag{9}$$

$$\psi \rho \Omega + \widehat{\alpha} \Omega \subset \rho \Omega. \tag{10}$$

To see this, by setting $\widehat{\gamma} = \min \gamma : \Lambda \subset \widehat{\gamma} \Omega$, consider the inclusions

$$\mathrm{e}^{-\beta t} \rho \Omega + \mathrm{e}^{-\widehat{\lambda} t} \widehat{\gamma} \Omega \subset \psi \mathrm{e}^{-\widehat{\lambda} t} \rho \Omega, \quad \forall t \ge \delta_m \tag{11}$$

$$\psi \rho \Omega + \widehat{\alpha} \Omega \subset \rho \Omega. \tag{12}$$

By construction, $\psi \in (0, 1)$ and $\rho > 0$ satisfying inclusions (11) and (12), satisfy (9) and (10), as well. On the other hand by algebraic manipulations of inclusions (11) and (12), it is easy to see that such parameters exist.

Let $\widehat{\rho}$ be a value of ρ such that (11) and (12) hold. Then the same inclusions hold for any $\rho \ge \widehat{\rho}$. Consider a control strategy which makes Ω a β-domain of attraction for $\mathcal{S}$. Given some j, assume by induction that

$$\xi(t_j, j) \in \mathrm{e}^{-\widehat{\lambda} t_j} \rho \Omega, \quad \rho \ge \widehat{\rho}.$$

Then any execution of $\mathcal{S}_d^c$ (the controlled switching system, with continuous disturbance) satisfies the condition

$$\xi(t, j) \in \mathrm{e}^{-\beta(t - t_j)} \mathrm{e}^{-\widehat{\lambda} t_j} \rho \Omega + \mathrm{e}^{-\widehat{\lambda}(t - t_j)} \mathrm{e}^{-\widehat{\lambda} t_j} \Lambda, \quad \forall t \in [t_j, t_j'],$$

and therefore, since $\widehat{\lambda} < \beta$ and $\Lambda \subset \widehat{\gamma} \Omega$,

$$\xi(t, j) \in \mathrm{e}^{-\widehat{\lambda} t} (\rho + \widehat{\gamma}) \Omega, \quad \forall t \in \left[t_j, t_j + \delta_m\right].$$

From (9),

$$\begin{aligned} \xi(t, j) &\in \mathrm{e}^{-\beta(t - t_j)} \mathrm{e}^{-\widehat{\lambda} t_j} \rho \Omega + \mathrm{e}^{-\widehat{\lambda}(t - t_j)} \mathrm{e}^{-\widehat{\lambda} t_j} \Lambda = \mathrm{e}^{-\widehat{\lambda} t_j} \left(\mathrm{e}^{-\beta(t - t_j)} \rho \Omega + \mathrm{e}^{-\widehat{\lambda}(t - t_j)} \Lambda \right) \\ &\subset \mathrm{e}^{-\widehat{\lambda} t_j} \left(\psi \mathrm{e}^{-\widehat{\lambda}(t - t_j)} \rho \Omega \right) = \psi \mathrm{e}^{-\widehat{\lambda} t} \rho \Omega, \quad \forall t \in [t_j + \delta_m, t_j']. \end{aligned}$$

Since Ω is a β-domain of attraction, then

$$(q(j+1), R(e_j) x(t_j', j)) \in \psi \mathrm{e}^{-\widehat{\lambda} t_{j+1}} \rho \Omega,$$

and since $\widehat{\lambda} \le \lambda$ and $(q\,(j+1)\,, F(e_j) w(t_j')) \in \widehat{\alpha} \mathrm{e}^{-\lambda t_j'} \Omega \subset \widehat{\alpha} \mathrm{e}^{-\widehat{\lambda} t_{j+1}} \Omega$, then from (10),

$$(q(j+1), R(e_j) x(t_j', j) + F(e_j) w(t_j')) \in \rho \mathrm{e}^{-\widehat{\lambda} t_{j+1}} \Omega.$$

Then by induction we can argue that any execution of $\mathcal{S}_d^c$ with hybrid initial state in $\rho \Omega$, $\rho \ge \widehat{\rho}$, satisfies the conditions:

$$\xi(t, j) \in \mathrm{e}^{-\widehat{\lambda} t} (\rho + \widehat{\gamma}) \Omega, \quad \forall t \in \left[t_j, t_j + \delta_m\right], \forall j = 0, 1, \ldots$$

$$\xi(t, j) \in \mathrm{e}^{-\widehat{\lambda} t} \rho \Omega, \quad \forall t \in [t_j + \delta_m, t_j'], \forall j = 0, 1, \ldots$$

and hence the above condition implies that, if $\xi(0,0) \in \rho\Omega$, $\rho \geq \widehat{\rho}$

$$\xi(t,j) \in \mathrm{e}^{-\widehat{\lambda}t}\left(\rho + \widehat{\gamma}\right)\Omega, \quad \forall t \in [t_j, t'_j],\ \forall j = 0, 1, \ldots.$$

If $\xi(0,0) \in \rho\Omega$, $\rho \in [0, \widehat{\rho})$ then

$$\xi(t,j) \in \mathrm{e}^{-\widehat{\lambda}t}\left(\widehat{\rho} + \widehat{\gamma}\right)\Omega, \quad \forall t \in [t_j, t'_j],\ \forall j = 0, 1, \ldots.$$

Finally, if $\xi(0,0) \in \mu\rho\mathcal{B}$, μ such that $\Omega \subset \mu\mathcal{B}$, $\rho \geq \widehat{\rho}$, then

$$\xi(t,j) \in \mu\mathrm{e}^{-\widehat{\lambda}t}\left(\rho + \widehat{\gamma}\right)\mathcal{B}, \quad \forall t \in [t_j, t'_j],\ \forall j = 0, 1, \ldots$$

and if $\xi(0,0) \in \mu\rho\mathcal{B}$, μ such that $\Omega \subset \mu\mathcal{B}$, $\rho \in [0, \widehat{\rho})$, then

$$\xi(t,j) \in \mu\mathrm{e}^{-\widehat{\lambda}t}\left(\widehat{\rho} + \widehat{\gamma}\right)\mathcal{B}, \quad \forall t \in [t_j, t'_j],\ \forall j = 0, 1, \ldots.$$

Finally by setting $\mu_0 = \mu\rho$ and $\gamma = \widehat{\rho} + \widehat{\gamma}$ the statement holds.

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