Invariance in Stochastic Dynamical Control Systems

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***Abstract*— Aim of this paper is to set invariance in stochastic dynamical control systems. Given a set within which the state of the dynamical system should evolve, we study conditions for finding a control strategy that maximizes the probability for the state to be in the given set within a fixed a–priori finite time horizon. We formulate an optimal control problem and we solve the problem at hand by using a dynamic programming approach. Some results towards a generalization of these results to the case of infinite–time horizon are also derived.**

***Keywords*— Stochastic dynamical control system, Stochastic Invariance Problem, Dynamic Programming, Optimal Control.**

I. INTRODUCTION

In the last few years, several stochastic hybrid models have been proposed and studied in the literature (see [15], [4] for an overview), because of their capability to capture non–smooth phenomena arising in real life complex systems such as Air Traffic Management (ATM) [4]. A general model of stochastic hybrid systems has been presented in [6], which includes as special cases many stochastic hybrid systems available in the literature as for example Piecewise Deterministic Markov Processes [7], Switching Diffusion Processes [9] and Stochastic Hybrid Systems [10] (see [15] for a formal comparison on these models). However, the analysis of the dynamical properties of these sophisticated models is very difficult. Preliminary work on the analysis of stochastic hybrid system properties was given in [5] and [6], where reachability problems of Piecewise Deterministic Markov Processes and of General Stochastic Hybrid Systems were formulated. Reachability problems for stochastic dynamical systems arise for example in a conflict resolution problem in a *free flight* configuration, one of the most challenging problems in the context of ATM (see e.g. [11]). Unfortunately, results reported in [5] and [6] do not include computationally feasible methodologies for approaching this kind of reachability problem. In [13], the conflict resolution problem is addressed with a Monte Carlo simulation based approach. In [1], some reachability problems for discrete–time stochastic hybrid systems are considered and solved.

In this paper, we present a first contribution towards the development of a computationally feasible solution to reachability problems for stochastic dynamical control systems. We consider non–linear discrete–time stochastic dynamical control systems and formulate for this class of systems the *Stochastic Invariance Problem*: Given a set, within which the state of the dynamical system must evolve, we derive conditions for finding a control strategy that maximizes the probability for the state to be in the given target set, within a fixed a–priori finite time horizon. The general formulation of the Stochastic Invariance Problem provides a way for addressing reachability problems as a special case (i.e. finding an optimal control strategy guaranteeing to reach a target set at a given time instant with maximal probability). Moreover, some new definitions of invariant and contractive sets are introduced, which generalize to the stochastic setting well–known notions available in the context of deterministic dynamical systems (e.g. [3]). To solve the Stochastic Invariance Problem, we use a dynamic programming approach [2]. In particular, we reformulate the Stochastic Invariance Problem as an optimal control problem and solve the optimal control problem using a dynamic programming approach. A generalization of the results in finite–time horizon to the case of infinite–time horizon is also described. A specialization of the results to the context of one dimensional stochastic control affine systems is presented as an example of application of the proposed methodology.

The methodology proposed in this paper has been applied in [16] to a quantitative finance problem. In particular [16] sets a unified framework to treat optimal dynamic assets allocation, and generalizes classical Markowitz Portfolio approach [14] to multi–period investments and non–gaussian hypotheses on asset classes performance stochastic dynamics.

This paper is organized as follows. Section 2 is devoted to the formal definition of the Stochastic Invariance Problem. In Section 3, we solve the Stochastic Invariance Problem. In Section 4 we address the special case of one dimensional stochastic control affine systems with gaussian noise. Section 5 offers some concluding remarks.

II. PRELIMINARIES AND PROBLEM STATEMENT

This section is devoted to the formal definition of the class of systems and of the control problem that we address in this paper.

Consider the following non–linear stochastic discrete–time dynamical control system:

$$x_{k+1} = f(x_k, u_k, W_k, k), \tag{1}$$

This work has been partially supported by the HYCON Network of Excellence, contract number FP6-IST-511368 and by Ministero dell'Istruzione, dell'Universita' e della Ricerca under Projects MACSI and SCEF (PRIN05).

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where, for any $k \in \mathbb{N}$:

- $x_k \in \mathbf{X} \subset \mathbb{R}^n$ is the state and $\mathbf{X}$ is the state space;
- $u_k \in \mathbf{U} \subset \mathbb{R}^m$ is the control input and $\mathbf{U}$ is the control input space;
- $W_k \in \mathbf{W} \subset \mathbb{R}^l$ is a random variable with probability density function $p_{W_k}, \forall k \in \mathbb{N}$ and $\mathbf{W}$ is the noise space;
- $f : \mathbf{X} \times \mathbf{U} \times \mathbf{W} \times \mathbb{N} \to \mathbf{X}$ is the vector field.

We assume that x_0 is given by means of a random variable X_0, whose probability density function is p_{X_0} and that random variables X_0, W_k, $\forall k \in \mathbb{N}$ are independent one each other. Given any $x \in \mathbf{X}$, $u \in \mathbf{U}$ and $k \in \mathbb{N}$, the probability density function of random variable $f(x, u, W_k, k)$ is denoted by $p_{f(x,u,W_k,k)}$.

We focus on the following class of controls μ:

$$\mathcal{U} = \{\mu : \mathbf{X} \times \mathbb{N} \to \mathbf{U}\},$$

namely, the class of time–varying feedback functions. Let $\mathcal{U}_N$ be the class of control inputs sequence $\pi = \{\mu_k\}_{k=0,1,...,N}$ such that $\mu_k \in \mathcal{U}$, for any $k = 0, 1, ..., N$. Any $\pi \in \mathcal{U}_N$ is called *control policy*. Given $N \in \mathbb{N}$, and $\pi = \left\{\mu_0, \mu_1, ..., \mu_{N-1}\right\} \in \mathcal{U}_{N-1}$, we set

$$\pi^k = \left\{\mu_k, \mu_{k+1}, ..., \mu_{N-1}\right\},$$

for any $k \in \mathbb{N}$; then $\pi^0 = \pi$.

In *safety critical problems* (e.g. [8] and the references therein), which arise in many engineering domains as Air Traffic Management [13], finance [16], etc., an important goal is to find an optimal control strategy guaranteeing to *reach* a given target set or to *remain* in a given target set, within a finite time horizon. The notions of reachable and invariant sets are therefore very important (see e.g. [3]) and can be extended to stochastic dynamical control systems, as indicated below.

Given a sequence of sets $\Sigma_k \subset \mathbf{X}$, $k = 0, 1, ...$ representing the sets of 'good' states within which the state evolution of system (1) must evolve, our problem is to find a control policy that maximizes the probability of the state x_k to be in Σ_k, for any time k within a finite time horizon N, i.e.

$$x(k) \in \Sigma_k, \forall k = 0, 1, ..., N.$$

More formally, let $(\Omega, \mathcal{F}, P)$ be the probability space associated with system (1).

Problem 1: (Stochastic Invariance Problem) Given a finite time horizon $N \in \mathbb{N}$ and a sequence of sets $\{\Sigma_k\}_{k=0,1,...,N}$, where for any $k = 0, 1, ..., N$, Σ_k are Borel subsets of $\mathbf{X}$, find the optimal control policy $\pi^* \in \mathcal{U}_{N-1}$ that maximizes

$$P(\{\omega \in \Omega : x_k \in \Sigma_k, \forall k = 0, 1, ..., N\}). \tag{2}$$

Denote by $p^*(N)$ the optimal value attained by (2). For notational simplicity, we set

$$\begin{aligned} &P(\{\omega \in \Omega : x_k \in \Sigma_k, \forall k = 0, 1, ..., N\}) = \\ &P(x_k \in \Sigma_k, \forall k = 0, 1, ..., N). \end{aligned} \tag{3}$$

The formulation of the Stochastic Invariance Problem is quite general to include as special cases also some other sub–problems as the classical reachability problem, invariance problem and the characterization of contractive sets in a stochastic fashion, thus generalizing classical notions, well–known in the context of deterministic systems (e.g. [3]) to the case of stochastic dynamical control systems, as shown in the following definition.

Definition 1: Given $N \in \mathbb{N}$ and a control policy $\pi \in \mathcal{U}_N$, a set $\Sigma \subset \mathbf{X}$ is said to be:

- *reachable in N steps* with probability $p \in [0, 1]$ if
$$P(x_N \in \Sigma, x_N \in \mathbf{X}, x_{N-1} \in \mathbf{X}, ..., x_0 \in \mathbf{X}) = p; \tag{4}$$
- *invariant in N steps* with probability $p \in [0, 1]$ if
$$P(x_k \in \Sigma, \forall k = 0, 1, ..., N) = p; \tag{5}$$
- *λ–contractive in N steps* with probability $p \in [0, 1]$, for some $\lambda \in (0, 1)$, if
$$P(x_k \in \lambda^k \Sigma, \forall k = 0, 1, ..., N) = p. \tag{6}$$

A set $\Sigma \subset \mathbf{X}$ is said to be *invariant*, with probability p if

$$\lim_{N \to \infty} P(x_k \in \Sigma, \forall k = 0, 1, ..., N) = p$$

and *λ–contractive*, with probability p if

$$\lim_{N \to \infty} P(x_k \in \lambda^k \Sigma, \forall k = 0, 1, ..., N) = p.$$

It is readily seen that the tasks of finding optimal control policies maximizing probability quantities (4), (5) or (6) can be cast into the framework of the Stochastic Invariance Problem, by appropriately defining the sequence of sets $\Sigma_k \subset \mathbf{X}$, $k = 0, 1, ..., N$.

III. MAIN RESULT

This section is devoted to the characterization of the Stochastic Invariance Problem. More precisely, we reformulate the Stochastic Invariance Problem as an optimal control problem and, inspired by [2], we solve the obtained optimal control problem by using a dynamic programming approach. Some results towards a generalization of the results in finite–time horizon to the case of infinite–time horizon are also illustrated.

The first result makes explicit in the probability quantity (2), the dependence on the dynamics of system (1) and on the statistics of the noise W_k, $k \in \mathbb{N}$ and of the initial state random variable X_0.

Lemma 1: Given $N \in \mathbb{N}$, a control policy $\pi = \{\mu_k\}_{k=0,...,N-1} \in \mathcal{U}_{N-1}$ and a sequence of sets $\{\Sigma_k\}_{k=0,...,N}$,

$$P(x_k \in \Sigma_k, k = 0, 1, ..., N) = \int_{\Sigma_N} \mathcal{I}(N, x) dx, \tag{7}$$

where for any $x \in \mathbf{X}$ and $k = 1, 2, ..., N$

$$\mathcal{I}(k, x) = \begin{cases} p_{X_0}(x), \quad k = 0, \\ \int\limits_{\Sigma_{k-1}} p_{f(z,\mu_{k-1},W_{k-1},k-1)}(x)\mathcal{I}(k-1, z)dz, \\ k = 1, 2, ..., N. \end{cases} \tag{8}$$

Proof: By induction. For $k = 0$,

$$P(x_0 \in \Sigma_0) = \int\limits_{\Sigma_0} p_{X_0}(z)dz = \int\limits_{\Sigma_0} \mathcal{I}(0, z)dz.$$

Suppose that (7) holds for step $k - 1$. By Bayes formula, the following chain of equalities holds:

$$\begin{aligned} & P(x_0 \in \Sigma_0,\ x_1 \in \Sigma_1,\ ...,\ x_k \in \Sigma_k) = \\ & = \int\limits_{\Sigma_{k-1}} P(x_k \in \Sigma_k \mid (x_{k-1} \in dx, x_{k-2} \in \Sigma_{k-2}, ...) \\ & \quad \cdot P(x_{k-1} \in dx, x_{k-2} \in \Sigma_{k-2}, ..., x_0 \in \Sigma_0) \\ & = \int\limits_{\Sigma_{k-1}} (\int\limits_{\Sigma_k} p_{f(x,\mu_{k-1},W_{k-1},k-1)}(z)dz)\mathcal{I}(k-1, x)dx \\ & = \int\limits_{\Sigma_k} (\int\limits_{\Sigma_{k-1}} p_{f(x,\mu_{k-1},W_{k-1},k-1)}(z)\mathcal{I}(k-1, x)dx)dz \\ & = \int\limits_{\Sigma_k} \mathcal{I}(k, z)dz. \end{aligned}$$

Thus, (7) holds for step k. ■

The following result shows that for any choice of the target sets sequence $\{\Sigma_k\}_{k=0,1,...,N}$ the probability to stay in $\{\Sigma_k\}_{k=0,1,...,N}$, is less than or equal to the probability to stay in the target set $\{\Sigma_k\}_{k=0,1,...,N-1}$.

Proposition 2: For any $N \in \mathbb{N}$, any control policy $\pi = \{\mu_k\}_{k=0,...,N-1} \in \mathcal{U}_{N-1}$ and any sequence of sets $\{\Sigma_k\}_{k=0,1,...,N}$,

$$P(x_k \in \Sigma_k, k = 0, ..., N) \leq P(x_k \in \Sigma_k, k = 0, ..., N-1).$$

Proof: By Lemma 1:

$$\begin{aligned} & P(x_k \in \Sigma_k, k = 0, 1, ..., N) = \int\limits_{\Sigma_N} \mathcal{I}(N, \Sigma_N)dz \\ & = \int\limits_{\Sigma_N} (\int\limits_{\Sigma_{N-1}} p_{f(x,\mu_{N-1},W_{N-1},N-1)}(z)\mathcal{I}(N-1, x)dx)dz \\ & = \int\limits_{\Sigma_{N-1}} (\int\limits_{\Sigma_N} p_{f(x,\mu_{N-1},W_{N-1},N-1)}(z)dz)\mathcal{I}(N-1, x)dx \\ & \leq \int\limits_{\Sigma_{N-1}} \mathcal{I}(N-1, x)dx \\ & = P(x_k \in \Sigma_k, k = 0, 1, ..., N-1), \end{aligned}$$

and hence the result follows. ■

Inspired by [2], we now define the cost function associated to the probability quantity (3) and hence to the Stochastic Invariance Problem. Given a sequence of sets Σ_k, $k = 0, 1, ...$, we introduce the following cost function V which associates a real number $V(k, x, \pi^k) \in [0, 1]$ to a triple (k, x, π^k) by:

$$V(k, x, \pi^k) = \begin{cases} I_{\Sigma_N}(x), \quad k = N; \\ \int\limits_{\Sigma_{k+1}} V(k+1, z, \pi^{k+1})p_{f(x,\mu_k,W_k,k)}(z)dz \\ k = 0, 1, ..., N-1, \end{cases} \tag{9}$$

where $I_A(x)$ is the indicator function of a Borel subset A of Σ_N, i.e.

$$I_A(x) = \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

We assume that all control policies μ are such that the cost function V as in (9), is well–defined. The following result establishes a formal functional relationship between V and $\mathcal{I}$ and hence between V and the probability quantity (2).

Proposition 3: Given $N \in \mathbb{N}$, a control policy $\pi \in \mathcal{U}_{N-1}$ and a sequence of sets $\{\Sigma_k\}_{k=0,1,...,N}$, for any $k = 0, 1, ..., N:$

$$\int\limits_{\Sigma_N} \mathcal{I}(N, x)dx = \int\limits_{\Sigma_k} V(k, x, \pi^k)\mathcal{I}(k, x)dx. \tag{10}$$

Proof: By induction. By definition of $\mathcal{I}$ in (8),

$$\begin{aligned} \int\limits_{\Sigma_N} \mathcal{I}(N, x)dx &= \int\limits_{\Sigma_N} I_{\Sigma_N}(x)\mathcal{I}(N, x)dx \\ &= \int\limits_{\Sigma_N} V(N, x, \pi^N)\mathcal{I}(N, x)dx. \end{aligned}$$

Hence, the statement holds for $k = N$. By proceeding backwards, we suppose that (10) is true for step k and we prove that (10) is true for step $k-1$. By replacing equation (8) into equation (10), and by equation (9), we have:

$$\begin{aligned} & \int\limits_{\Sigma_N} \mathcal{I}(N, x)dx = \\ & \int\limits_{\Sigma_k} V(k, x, \pi^k)(\int\limits_{\Sigma_{k-1}} p_{f(z,\mu_{k-1},W_{k-1},k-1)}(x) \cdot \\ & \mathcal{I}(k-1, z)dz)dx = \\ & \int\limits_{\Sigma_{k-1}} (\int\limits_{\Sigma_k} V(k, x, \pi^k)p_{f(z,\mu_{k-1},W_{k-1},k-1)}(x)dx) \cdot \\ & \mathcal{I}(k-1, z)dz = \\ & \int\limits_{\Sigma_{k-1}} V(k-1, x, \pi^{k-1})\mathcal{I}(k-1, z)dz, \end{aligned}$$

and hence the result follows. ■

The result above gives the way for rewriting the probability quantity (3) in terms of the cost function V:

Proposition 4: Given $N \in \mathbb{N}$, a control policy $\pi \in \mathcal{U}_{N-1}$ and a sequence of sets $\{\Sigma_k\}_{k=0,1,...,N}$,

$$P(x_k \in \Sigma, \forall k = 0, ..., N) = \int\limits_{\Sigma_0} V(0, x, \pi)p_{X_0}(x)dx.$$

Proof: By applying Proposition 3 at step $k = 0$, and by Lemma 1, the statement holds. ■

By Proposition 4, it is possible to rewrite the Stochastic Invariance Problem, as follows.

Problem 1: (Stochastic Invariance Problem) Given a finite time horizon $N \in N$ and a sequence of sets $\{\Sigma_k\}_{k=0,1,\ldots,N}$, where for any $k = 0,\ 1,\ \ldots,\ N$, Σ_k are Borel subsets of X, compute:

$$\pi^* = \arg\sup_{\pi} \int\limits_{\Sigma_0} V(0, x, \pi) p_{X_0}(x) dx. \tag{11}$$

Problem 1, as reformulated above, highlights connections between the Stochastic Invariance Problem and optimal control problems. Following [2], we now show how to compute an optimal control policy by using a dynamic programming approach.

Before giving the main result of this paper, we need the following technical result.

Lemma 5: Given a measurable subset Σ of $\mathbf{X}$, let be

$$\begin{aligned} &\eta_1 : \mathbf{X} \times \mathbf{U} \to \mathbb{R}, \\ &\eta_2 : \mathbf{X} \to \mathbb{R}, \end{aligned}$$

such that $\eta_2(x) \geq 0$ for any $x \in \Sigma$ and let $\mathcal{U}_0$ be the class of feedback functions, i.e. $\mathcal{U}_0 = \{\mu : \mathbf{X} \to \mathbf{U}\}$. The optimal control policy $\mu^* \in \mathcal{U}_0$, solving the following optimization problem:

$$\sup_{\mu \in \mathcal{U}_0} \int\limits_{\Sigma} \eta_1(x, \mu(x)) \eta_2(x) dx, \tag{12}$$

is such that for any $\mu \in \mathcal{U}_0$,

$$\eta_1(x, \mu(x)) \leq \eta_1(x, \mu^*(x)), \tag{13}$$

almost everywhere with respect to $x \in \Sigma$. Conversely, if $\mu^* \in \mathcal{U}_0$ satisfies (13) for any $x \in \Sigma$, then μ^* is the solution to the optimization problem (12).

Proof: For the sake of contradiction, suppose that there exists a measurable set $A \subset \Sigma$, with non–zero Lebesgue measure, such that for any $x \in A$, inequality (13) is not true. Then, there exists a feedback control policy $\bar{\mu} \in \mathcal{U}_0$ such that

$$\int\limits_{A} \eta_1(x, \bar{\mu}(x)) \eta_2(x) dx > \int\limits_{A} \eta_1(x, \mu^*(x)) \eta_2(x) dx.$$

Define the following control policy,

$$\hat{\mu}(x) = \begin{cases} \mu^*(x), & \text{if } x \in \Sigma \backslash A, \\ \bar{\mu}(x), & \text{if } x \in A. \end{cases}$$

Then,

$$\begin{aligned} &\int\limits_{x \in \Sigma} \eta_1(x, \hat{\mu}(x)) \eta_2(x) dx \\ &= \int\limits_{\Sigma \backslash A} \eta_1(x, \mu^*(x)) \eta_2(x) dx + \int\limits_{A} \eta_1(x, \bar{\mu}(x)) \eta_2(x) dx \\ &> \int\limits_{\Sigma \backslash A} \eta_1(x, \mu^*(x)) \eta_2(x) dx + \int\limits_{A} \eta_1(x, \mu^*(x)) \eta_2(x) dx \\ &= \int\limits_{\Sigma} \eta_1(x, \mu^*(x)) \eta_2(x) dx, \end{aligned}$$

and hence μ^* is not the optimal control policy that solves problem (12). The second part of the statement is trivial. ■

We now give the main result of the paper: an algorithm that enables the computation of optimal control policy π^* solving the Stochastic Invariance Problem, or equivalently equation (11).

Theorem 6: The optimal value of the Stochastic Invariance Problem is equal to

$$p^*(N) = \int_{\Sigma_0} J_0(x) p_{X_0}(x) dx,$$

where $J_0(x)$ is given by the last step of the following algorithm,

$$\begin{aligned} &J_N(x) = I_{\Sigma_N}(x), \\ &J_k(x) = \sup_{u_k \in \mathbf{U}} \int\limits_{\Sigma_{k+1}} J_{k+1}(z) p_{f(x, u_k, W_k, k)}(z) dz, \qquad (14) \\ &k = N-1, N-2, \ldots, 0. \end{aligned}$$

Furthermore, if $\hat{\mu}_k(x) = \hat{u}_k$ maximizes the right hand side of Equation (14) for each $x \in \Sigma_k$ and $k \in \mathbb{N}$, then the class of policies $\hat{\pi} = \left\{\hat{\mu}_0, \ldots, \hat{\mu}_{N-1}\right\}$ is optimal.

Proof: For $k = 0, 1, \ldots, N-2$, let $J_k^*(x)$ be the optimal cost for the $(N-k)$–stage problem that starts at state x and time k, and ends at time N, i.e. $J_k^*(x) = \sup_{\pi^k} V(k, x, \pi^k)$. For $k = N$, we define $J_N^*(x) = I_\Sigma(x)$. We will show by induction that the functions $J_k^*(.)$ are equal to the functions $J_k(.)$, as defined in (14), so that for $k = 0$ we obtain the desired result. Assume that for some k and all x, we have that $J_{k+1}^*(x) = J_{k+1}(x)$. Then, since $\pi^k = \left(\mu_k, \pi^{k+1}\right)$, we have for all x,

$$\begin{aligned} &J_k^*(x) = \sup_{(\mu_k, \pi^{k+1})} V(k, x, \pi^k) = \\ &\sup_{(\mu_k, \pi^{k+1})} \int\limits_{\Sigma_{k+1}} V(k+1, z, \pi^{k+1}) p_{f(x, \mu_k, W_k, k)}(z) dz = \\ &\sup_{\mu_k} \int\limits_{\Sigma_{k+1}} (\sup_{\pi^{k+1}} V(k+1, z, \pi^{k+1})) p_{f(x, \mu_k, W_k, k)}(z) dz = \\ &\sup_{\mu_k} \int\limits_{\Sigma_{k+1}} J_{k+1}^*(z) p_{f(x, \mu_k, W_k, k)}(z) dz = \\ &\sup_{\mu_k} \int\limits_{\Sigma_{k+1}} J_{k+1}(z) p_{f(x, \mu_k, W_k, k)}(z) dz = \\ &\sup_{u_k \in \mathbf{U}} \int\limits_{\Sigma_{k+1}} J_{k+1}(z) p_{f(x, u_k, W_k, k)}(z) dz = J_k(x), \end{aligned}$$

completing the induction. The second equality holds by definition of V in (9). In the third equality, we moved the supremum over π^{k+1} inside the integral because of Lemma 5 and of the principle of optimality argument (see e.g. [2].) In the forth equality, we used the definition of $J_{k+1}^*(x)$, and in the fifth equality we used the induction hypothesis. Finally, in the sixth equality, we converted the supremum over μ_k to a supremum over u_k, using the fact that for any function F of z and u, we have:

$$\sup_{\mu \in \mathcal{U}_1} F(z, \mu(z)) = \sup_{u \in \mathbf{U}} F(z, u).$$

■

Remark 1: Note that if the input space $\mathbf{U}$ is compact, then equation (14) can be replaced by:

$$J_k(x) = \max_{u_k \in U} \int_{\Sigma_{k+1}} J_{k+1}(z) p_{f(x,u_k,W_k,k)}(z) dz.$$

Stochastic Invariance Problem as formalized in Problem 1, is given in a finite time horizon. It can also be of interest to study an infinite–time horizon version of the Stochastic Invariance Problem. We conclude this section by showing that the Stochastic Invariance Problem in infinite–time horizon is well–defined and that a solution always exists. More precisely, we show that given a sequence of sets $\{\Sigma_k\}_{k\in\mathbb{N}}$, where Σ_k is a Borel subset of $\mathbf{X}$, for any $k \in \mathbb{N}$, the following limit does exist:

$$\begin{aligned} p^* &= \lim_{N\to\infty} p^*(N) \\ &= \lim_{N\to\infty} \sup_{\pi} P(x_k \in \Sigma_k, \forall k = 0, ..., N). \end{aligned}$$

The following result shows that $p^*(N)$ is a decreasing function of $N \in \mathbb{N}$.

Proposition 7: Given a sequence of sets $\{\Sigma_k\}_{k\in\mathbb{N}}$, where Σ_k is a Borel subset of $\mathbf{X}$ for any $k \in \mathbb{N}$,

$$p^*(N+1) \leq p^*(N), \qquad \forall N \in \mathbb{N}.$$

Proof: Suppose by contradiction that there exists $N \in \mathbb{N}$ such that:

$$p^*(N) < p^*(N+1). \tag{15}$$

Let $\pi_N^* = (\mu, \pi)$ be the optimal control policy that solves the Stochastic Invariance Problem for step $N+1$. By Proposition 2,

$$p^*(N+1) = \int\limits_{\Sigma_{N+1}} \mathcal{I}(N+1, x) dx \leq \int\limits_{\Sigma_N} \mathcal{I}(N, x) dx, \tag{16}$$

where $\mathcal{I}(N+1, x)$ is computed w.r.t. the control policy π_N^* and $\mathcal{I}(N, x)$ is computed w.r.t. the control policy π. By combining inequalities (15) and (16), one gets:

$$p^*(N) < p^*(N+1) \leq \int\limits_{\Sigma_N} \mathcal{I}(N, x) dx,$$

and therefore $p^*(N)$ is not the solution to the Stochastic Invariance Problem at step N. ■

As a consequence, since the sequence $\{p^*(N)\}_{N\in\mathbb{N}}$ is lower bounded by 0, we conclude that:

Theorem 8: For any sequence of sets $\{\Sigma_k\}_{k\in\mathbb{N}}$, where Σ_k is a Borel subset of $\mathbf{X}$ for any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} p^*(N),$$

does exist.

IV. A SPECIAL CASE: ONE DIMENSIONAL STOCHASTIC CONTROL AFFINE SYSTEMS

In this section, we consider the Stochastic Invariance Problem for the class of one dimensional control affine non–linear systems (e.g. [12]), with gaussian noises. More precisely, consider the following system:

$$\begin{cases} x_{k+1} = f_1(x_k, k) + f_2(x_k, k) u_k + W_k, \\ x_k \in \mathbf{X},\ u_k \in \mathbf{U},\, W_k \in \mathbf{W},\, \forall k \in \mathbb{N}, \end{cases} \tag{17}$$

where $\mathbf{X} = \mathbf{U} = \mathbf{W} = \mathbb{R}$ and X_0 and W_k are gaussian random variables with statistics $\mathcal{N}(m, \sigma)$ and $\mathcal{N}(m_k, \sigma_k)$, for any $k \in \mathbb{N}$. We suppose that Lebesgue measure of the set $\{x \in \mathbb{R} : f_2(x, .) = 0\}$ is zero.

Consider a finite time horizon $N \in \mathbb{N}$ and a sequence of sets

$$\Sigma_k = [a_k, b_k], \quad k = 0, 1, ..., N, \tag{18}$$

where $a_k < b_k < \infty$, for any $k = 0, 1, ..., N$.

We now characterize the Stochastic Invariance Problem for system (17), sequence of sets (18), within a finite time horizon N.

For any triple (x, u, k), the random variable

$$f_1(x, k) + f_2(x, k) u + W_k,$$

is gaussian with statistics

$$\mathcal{N}(f_1(x, k) + f_2(x, k) u + m_k, \sigma_k).$$

The cost function is given by:

$$V(k, x, \pi^k) = \begin{cases} I_{[a_n, b_n]}(x), \quad k = N; \\ \\ \int_{a_{k+1}}^{b_{k+1}} V(k+1, z, \pi^{k+1}) \cdot \\ p_{W_k}(w - f_1(x_k, k) - f_2(x_k, k) \mu_k) dw \\ \\ k = 0, 1, ..., N-1. \end{cases}$$

Consider the first step of the algorithm of Theorem 6:

$$J_{N-1}(x_{N-1}) = \sup_{\mu_{N-1} \in \mathcal{U}} V(N-1, x_{N-1}, \mu_{N-1}) \tag{19}$$

where:

$$\begin{aligned} &V(N-1, x_{N-1}, \mu_{N-1}) = \\ &\int_{a_N}^{b_N} p_{W_{N-1}}(w - f_1(x_{N-1}, N-1) + \\ &-f_2(x_{N-1}, N-1) \mu_{N-1}) dw. \end{aligned} \tag{20}$$

The control input μ_{N-1}^* that solves optimization problem (19) is given by:

$$\mu_{N-1}^*(x_{N-1}) = \begin{cases} \text{any, if } f_2(x_{N-1}, N-1) = 0; \\ \\ \frac{-m_{N-1} + \frac{a_N + b_N}{2} - f_1(x_{N-1}, N-1)}{f_2(x_{N-1}, N-1)}, \\ \text{otherwise.} \end{cases} \tag{21}$$

This completes step $N-1$ of the algorithm of Theorem 6. Set:

$$\begin{aligned} &\alpha_{N-1} = V(N-1, x_{N-1}, \mu_{N-1}^*) = \\ &\int_{-\frac{b_N - a_N}{2}}^{\frac{b_N - a_N}{2}} \frac{1}{\sqrt{2\pi}\sigma_{N-1}} \exp\left(-\frac{1}{2\sigma_{N-1}^2} z^2\right) dz \in (0, 1). \end{aligned}$$

An interpretation of the control μ_{N-1}^* in (21) is that it translates the mean value of the random variable $f_1(x_{N-1}, N-1) + f_2(x_{N-1}, N-1)\mu_{N-1} + W_{N-1}$ into $\frac{a_N + b_N}{2}$ and hence, by simple geometrical considerations, it maximizes $V(N-1, x_{N-1}, \mu_{N-1})$. Moreover, in this case, the obtained $J_{N-1}(x_{N-1})$ does not depend on x_{N-1}, and this makes the backward steps very easy to be computed. In fact, by proceeding backwards it is easy to see that for any $k = 0, 1, ..., N-2$:

$$\mu_k^*(x_k) = \begin{cases} \text{any, if } f_2(x_k, k) = 0 \\ \frac{-m_k + \frac{a_{k+1}+b_{k+1}}{2} - f_1(x_k,k)}{f_2(x_k,k)}, \text{ otherwise.} \end{cases}$$

Therefore the optimal value obtained for the Stochastic Invariance Problem is given by:

$$p^*(N) = \left(\prod_{k=0}^{N-1} \alpha_k \right) \int_{a_0}^{b_0} \exp\left(-\frac{1}{2\sigma} (x_0 - m)^2 \right) dx_0,$$

where:

$$\alpha_k = \int_{-\frac{b_{k+1}-a_{k+1}}{2}}^{\frac{b_{k+1}-a_{k+1}}{2}} \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left(-\frac{1}{2\sigma_k^2} z^2 \right) dz.$$

Consider now an infinite sequence of sets $\{\Sigma_k\}_{k\in\mathbb{N}}$, where $\Sigma_k = [a, b]$, for any $k \in \mathbb{N}$ and suppose that W_k are gaussian random variables with statistics $\mathcal{N}(m', \sigma')$ for any $k \in \mathbb{N}$. By setting:

$$\alpha = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{2\pi}\sigma'} \exp\left(-\frac{1}{2\sigma'^2} z^2 \right) dz \in (0, 1),$$

we have:

$$p^*(N) = \alpha^{N-1} \int_a^b \exp\left(-\frac{1}{2\sigma} (x_0 - m)^2 \right) dx_0.$$

Hence, the probability to remain in $[a, b]$ for any time $k \in \mathbb{N}$ is given by:

$$p^* = \lim_{N\to\infty} p^*(N) = 0.$$

V. CONCLUSION

In this paper, we formulated the Stochastic Invariance Problem (SIP) for dynamical stochastic control systems. Given a set within which the state of the dynamical system must evolve, we derived conditions for finding a control policy that maximizes the probability for the state to be in the given set within a fixed a–priori finite time horizon. We formulated the SIP as an optimal control problem and we solved this by using a dynamic programming approach [2]. We derived some results showing that the Stochastic Invariance Problem is well–defined in an infinite time horizon. Future work will concentrate on the study of the Stochastic Invariance Problem in an infinite–time horizon settings.

Acknowledgement: The first author wishes to thank C. Manes, P. Palumbo and P. Pepe for useful discussions on stochastic systems and an anonymous reviewer whose suggestions contributed to improve the paper.

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