Stabilizability of affine switching systems: A Kalman-like approach

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ABSTRACT. *In this paper, we focus on a particular subclass of hybrid systems, the class of affine switching systems. We propose hybrid state space decompositions, based on hybrid invariant subspaces, which reduce the computational effort required for checking the structural property of asymptotic stabilizability.*

RÉSUMÉ. *A définir par la commande*

KEYWORDS: *hybrid systems, stabilizability, Kalman decomposition.*

MOTS-CLÉS : *A définir par la commande*

1. Introduction

In this paper, we focus on affine switching systems [De 04a], a subclass of hybrid systems, where the continuous dynamics and the reset functions are affine and the transitions depend only on an event that acts as a discrete disturbance. The continuous dynamics are given by an affine dynamical control system (whose dynamical matrices depend on the current discrete state) and therefore an input function can be designed for controlling purposes.

Stability issues of hybrid systems have been extensively investigated in the last years (see e.g. in [Bra 98], [YE 98], [LIB 03], [SUN 05] and references therein). However checking stabilizability of switching systems is not an easy task in general (see e.g. [LIB 03]) and a complete characterization of stabilizability properties of switching systems is still missing. This is the reason why in this paper we focus on some structural reductions of the hybrid state space, which allow the original problem to be split into simpler subproblems. At first it is shown that asymptotic stabilizability of an affine switching system can be reduced to the asymptotic stabilizability of a linear switching system. Connections to the well–known Kalman decomposition of linear dynamical control systems are also established. The present paper extends the results of [De 06d]. Dual results on detectability based state space reductions have been recently established in a companion paper [De 06b].

The organization of the paper is as follows. We first recall some definitions of switching systems and stabilizability in Section 2. Then we define in Section 3 invariant hybrid subspaces, thereby extending to the hybrid framework the notions given in [BAS 69] for the linear case, and we propose an algorithm for the computation of the minimal invariant hybrid subspace containing a given hybrid subspace. In Section 4, by means of this minimal hybrid subspace, we define a state space transformation of the system, which allows stating conditions for stabilizability. Based on this result and on [De 06a], the given system is decoupled into controlled and autonomous linear switching subsystems. The asymptotic stabilizability of the first ones and the asymptotic stability of the latter ones imply the asymptotic stabilizability of the given system. Some concluding remarks are offered in Section 5.

2. Preliminaries and basic definitions

Aim of this section is to introduce the preliminary definitions and the problem setting of this paper: Section 2.1 introduces the class of affine switching systems and Section 2.2 formalizes the structural dynamical property under study.

2.1. *Switching systems*

In this section, we formally introduce the class of *affine switching systems*, following the general model of hybrid automata (see e.g. [Lyg 96]).

The hybrid state ξ of an affine switching system is composed of two components: the discrete state q_i, belonging to a finite set Q and the continuous state x, belonging to a linear space $\mathbb{R}^{n_i}$, whose dimension n_i depends on q_i. The evolution of the discrete state is governed by a Finite State Machine (FSM); a transition $e = (q_i, \sigma, q_h)$ may occur at time t from the discrete state q_i to the discrete state q_h, if the discrete disturbance σ occurs at time t. The evolution of the continuous state is described by a set of affine dynamical systems, whose matrices depend on the current discrete state q_i. Whenever a transition e occurs, the continuous state x is instantly reset to a new value $R(e)x + r(e)$, where $R(e)$ and $r(e)$ depend on the transition e. More formally,

Definition 2.1. *An affine switching system $\mathcal{S}$ is a tuple*

$$(\Xi, \Theta, S, E, M),$$

where:

– $\Xi = \bigcup_{q_i \in Q} \{q_i\} \times \mathbb{R}^{n_i}$ is the hybrid state space, where $Q = \{q_i, i \in J\}$ is the set of discrete states, $J = \{1, 2, \ldots, N\}$ and $\mathbb{R}^{n_i}$ is the continuous state space associated with the discrete state $q_i \in Q$;

– $\Theta = \Sigma \times U$ is the hybrid input space, where $\Sigma = \{\sigma_h, h \in J_1\}$ is the set of discrete disturbances, $J_1 = \{1, 2, \ldots, N_1\}$ and $U = \mathbb{R}^m$ is the continuous input space;

– S is a map associating to any discrete state $q_i \in Q$ the following affine dynamical control system:

$$\dot{x}(t) = A_i x(t) + B_i u(t) + d_i,$$

where $x(t) \in \mathbb{R}^{n_i}$ is the continuous state, and $u(t)$ is the continuous input;

– $E \subset Q \times \Sigma \times Q$ is a collection of transitions;

– M is a function that associates to any $e = (q_i, \sigma, q_h) \in E$ and any $x \in \mathbb{R}^{n_i}$ the state $M(e, x) = R(e)x + r(e) \in \mathbb{R}^{n_h}$.

An affine switching system is said to be a *linear* switching system if $d_i = 0$, $\forall q_i \in Q$ and $r(e) = 0$, $\forall e \in E$ and for simplicity we refer to a linear switching system by means of the tuple (Ξ, Θ, S, E, R). An affine switching system is said to be *autonomous* if $U = \{0\}$.

We now formally define the semantics of affine switching systems. First of all we assume throughout the paper that *the discrete disturbance is not available for measurements*, thus yielding a non–deterministic system, and that the class of admissible continuous inputs is the set $\mathcal{U}$ of piecewise continuous control functions $u : \mathbb{R} \rightarrow U$. As defined in [Lyg 96], a *hybrid time basis* τ is an infinite or finite sequence of sets $I_j = \{t \in \mathbb{R} : t_j \leq t \leq t'_j\}$, with $t'_j = t_{j+1}$; set $card(\tau) = L + 1$. If $L < \infty$, then t'_L can be finite or infinite. A hybrid time basis τ is said to be *finite*, if $L < \infty$ and $t'_L < \infty$ and *infinite*, otherwise. Since linear switching systems are time invariant, we assume without loss of generality that $t_0 = 0$ in any hybrid time basis. Throughout the paper, we assume that there is a minimum time separation between two consecutive transitions:

Assumption 1 (*Minimum dwell time*) There exists a real $\delta_m > 0$, called minimum dwell time [Mor 96], such that $t'_j - t_j \geq \delta_m$ for any hybrid time basis τ.

The existence of a minimum dwell time is a widely used assumption in the analysis of switching systems (e.g. [Mor 96], [LIB 03], [De 06c] and the references therein), and models the inertia of the system to react to an external (discrete) input. $\mathcal{T}$ denotes the set of all hybrid time bases satisfying Assumption 1. The temporal evolution of an affine switching system can be defined as follows.

Definition 2.2. *(Switching system execution) An execution χ of an affine switching system $\mathcal{S}$ is a collection $(\xi_0, \tau, \sigma, u, \xi)$ with $\xi_0 \in \Xi$, $\tau \in \mathcal{T}$, $\sigma : \mathbb{N} \rightarrow \Sigma$, $u \in \mathcal{U}$, $\xi : \mathbb{R} \times \mathbb{N} \rightarrow \Xi$. The hybrid state evolution ξ is defined as follows:*

$$\begin{array}{ll} \xi\left(0,0\right) = \xi_0, & \\ \xi\left(t,j\right) = \left(q\left(j\right), x(t,j)\right), & t \in I_j, j = 0,1,...,L, \\ \xi\left(t_{j+1}, j+1\right) = \left(q\left(j+1\right), R(e_j)x(t'_j,j) + r(e_j)\right), & j = 0,1,...,L, \end{array}$$

where $q : \mathbb{N} \rightarrow Q$, $e_j = \left(q\left(j\right), \sigma\left(j\right), q\left(j+1\right)\right) \in E$ and $x\left(t,j\right)$ is the solution at time t of the dynamical system $S\left(q\left(j\right)\right)$, with initial time t_j, initial condition $x\left(t_j, j\right)$ and continuous input u.

Given $\mathcal{S}$ and an execution χ, set $\eta\left(t\right) = \xi\left(t,j\right)$, $t \in [t_j, t'_j)$, $j = 0,1,...,L$. We assume that *the hybrid state evolution is available for control synthesis*: the set

$$\mathcal{Y} = \{\eta|_{[0,t]}, \eta : \mathbb{R} \rightarrow \Xi, t \geq 0\}$$

embeds all the information on the hybrid state evolution available for control purposes. A *control strategy* φ is a function $\varphi : \mathcal{Y} \rightarrow U$ such that the function defined by $u\left(t\right) = \varphi(\eta|_{[0,t]})$, $t \geq 0$ belongs to $\mathcal{U}$. A switching system $\mathcal{S}$ together with a control strategy φ is called *controlled switching system* and its executions with $u\left(t\right) = \varphi(\eta|_{[0,t]})$, $t \geq 0$ are called *controlled executions.*

2.2. *Asymptotic stabilizability*

This section is devoted to the formal definition of asymptotic stabilizability of affine switching systems.

We start by formally defining the notion of equilibrium points for switching systems.

Definition 2.3. *An equilibrium point of an affine switching system $\mathcal{S}$ is a set*

$$\hat{\Xi} = \bigcup_{q_i \in Q} \{q_i\} \times \{x_i\}, \tag{1}$$

such that for any execution $\chi = (\xi_0, \tau, \sigma, u, \xi)$ of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \hat{\Xi}$ and control $u(t) = 0, \forall t \geq 0$,

$$\xi(t,j) \in \hat{\Xi}, \forall t \in I_j, \ \forall j = 0,...,L. \tag{2}$$

A set $\hat{\Xi}$ as in (1) is a controlled equilibrium point of an affine switching system $\mathcal{S}$ if there exists $\{u_i \in \mathbb{R}^m, i \in J\}$ such that condition (2) is satisfied for any execution $\chi = (\xi_0, \tau, \sigma, u, \xi)$ of $\mathcal{S}$ with hybrid initial state $\xi_0 \in \hat{\Xi}$ and control $u \in \mathcal{U}$ defined by:

$$u(t) = u_i, \quad q(j) = q_i, \quad t \in I_j, \quad j = 0, 1, ..., N.$$

By definition above, given an affine switching system $\mathcal{S}$, $\hat{\Xi}$ is a controlled equilibrium point for $\mathcal{S}$ if and only if there exists $\{u_i \in \mathbb{R}^m, i \in J\}$ such that the following equations are satisfied:

$$\begin{cases} A_i x_i + B_i u_i + d_i = 0, & \forall i = 1, 2, ..., N, \\ R(e) x_i + r(e) = x_h, & \forall e = (q_i, \sigma, q_h) \in E, \end{cases} \tag{3}$$

The set of equilibrium points of $\mathcal{S}$ as in (1), can be computed by solving linear equations (3) where $u_i = 0, \forall q_i \in Q$.

Given a linear switching system $\mathcal{S}$, the set:

$$0_{\mathcal{S}} = \bigcup_{q_i \in Q} \{q_i\} \times \{0_i\}, \tag{4}$$

is an equilibrium point for $\mathcal{S}$. In the following we refer to the set $0_{\mathcal{S}}$, as the *hybrid origin* associated with $\mathcal{S}$.

We can now formally introduce the notion of asymptotic stabilizability. Set

$$\mathcal{B} := \bigcup_{q_i \in Q} \{q_i\} \times \mathcal{B}_i,$$

where $\mathcal{B}_i = \{x \in \mathbb{R}^{n_i} : \|x\|_{n_i} \leq 1\}$ for any $i \in J$ and for any $\varepsilon \geq 0$, set

$$\varepsilon\mathcal{B} := \bigcup_{q_i \in Q} \{q_i\} \times \varepsilon\mathcal{B}_i.$$

Given an equilibrium point $\hat{\Xi}$ as in (1), define:

$$\hat{\Xi} + \varepsilon\mathcal{B} := \bigcup_{q_i \in Q} \{q_i\} \times (\{x_i\} + \varepsilon\mathcal{B}_i).$$

Definition 2.4. *Given an affine switching system $\mathcal{S}$, a controlled equilibrium point $\hat{\Xi}$ of $\mathcal{S}$ is asymptotically stabilizable if there exists a control strategy φ such that $\forall \varepsilon > 0$ and for all controlled executions of $\mathcal{S}$ with hybrid initial state in $\hat{\Xi} + \mathcal{B}$, there exists $\hat{t} > 0$ such that:*

$$\xi(t, j) \in \hat{\Xi} + \varepsilon\mathcal{B}, \forall t \in I_j \cap [\hat{t}, \infty), \ \forall j = \hat{\jmath}, \hat{\jmath} + 1, ..., L, \tag{5}$$

where $\hat{\jmath} = \min\{j : \hat{t} \in I_j\}$. The control strategy φ is called stabilizing for $\hat{\Xi}$.

We now show that, as in the case of affine and linear systems, asymptotic stabilizability of affine switching systems can be reduced to asymptotic stabilizability of linear switching systems.

Given an affine switching system $\mathcal{S} = (\Xi, \Theta, S, E, M)$, and a controlled equilibrium point $\hat{\Xi}$ of $\mathcal{S}$ as in (1), define a hybrid state space transformation that associates to any hybrid state $(q_i, x) \in \Xi$ the hybrid state $(q_i, x - x_i) \in \Xi$. By applying this transformation to the affine switching system $\mathcal{S}$, one obtains the following linear switching system $\mathcal{S}' = (\Xi, \Theta, S', E, M')$, where for any $q_i \in Q$, $S(q_i)$ is given by dynamics:

$$\dot{x}(t) = A_i x(t) + B_i u(t),$$

and for any $e \in E$, $M'(e, x) = R(e)x$. By construction, we obtain

Proposition 2.1. *Given an affine switching system $\mathcal{S}$, a controlled equilibrium point $\hat{\Xi}$ of $\mathcal{S}$ is asymptotically stabilizable if and only if the hybrid origin $0_{\mathcal{S}'}$ of $\mathcal{S}'$ is asymptotically stabilizable. If φ is a stabilizing control strategy for $\hat{\Xi}$, then φ' is a stabilizing control strategy for the hybrid origin $0_{\mathcal{S}'}$, where:*

$$\varphi'(\eta|_{[0,t]}) = \varphi(\eta|_{[0,t]}) - u_i, t \geq 0,$$

and q_i is the discrete state component of $\eta|_{[0,t]}$, $t \geq 0$.

From the result above, given a linear switching system $\mathcal{S}'$, a controlled equilibrium point $\hat{\Xi}$ of $\mathcal{S}'$ is asymptotically stabilizable if and only if the hybrid origin $0_{\mathcal{S}'}$ is asymptotically stabilizable. For this reason there is no loss of generality in characterizing asymptotic stabilizability of the hybrid origin of a linear switching system when studying asymptotic stabilizability of a controlled equilibrium point for an affine switching system.

A linear switching system $\mathcal{S}$ is said to be *asymptotically stabilizable* if the hybrid origin $0_{\mathcal{S}}$ of $\mathcal{S}$ is asymptotically stabilizable. If a linear switching system $\mathcal{S}$ is asymptotically stabilizable and verifies conditions (5) with $\hat{\Xi} = 0_{\mathcal{S}}$ and $\varepsilon = 0$, then $\mathcal{S}$ is called *controllable*. An autonomous linear switching system that is asymptotically stabilizable is said to be asymptotically stable.

Remark 1. *From the definition above, it is easy to see that a linear switching system $\mathcal{S}$ with minimum dwell time $\delta_m > 0$ is controllable if and only if every linear system $S(q)$, $q \in Q$ is controllable.*

Since our purpose is to reduce the state space while preserving stabilizability (hence an asymptotic property), we consider only executions of infinite duration.

3. Invariant hybrid subspaces

Aim of this section is to introduce an invariant linear hybrid subspace that will be the basis upon which stabilizability analysis for linear switching systems can be performed.

The notion of invariant linear subspace for linear switching systems can be defined as follows.

Definition 3.1. *A set*

$$\Omega = \bigcup_{i\in J'} \{q_i\} \times \Omega_i \subset \Xi$$

is a hybrid linear subspace of Ξ*, if* $J' = J$ *and* Ω_i *is a linear subspace of* $\mathbb{R}^{n_i}$*, for any* $i \in J$*.*

For shortness, a hybrid linear subspace will be simply called subspace.

Definition 3.2. *Given a linear switching system* $\mathcal{S}$*, a set*

$$\Omega = \bigcup_{i\in J} \{q_i\} \times \Omega_i \subset \Xi,$$

is $\mathcal{S}-$*invariant if, for any initial hybrid state* $\xi_0 \in \Omega$ *and for any execution* $\chi = (\xi_0, \tau, \sigma, u, \xi)$ *with* $u(t) = 0, \forall t \geq 0$*,*

$$\xi(t, j) \in \Omega, \forall t \in I_j, \forall j = 0, 1, ..., L.$$

The following result gives a necessary and sufficient condition for a subspace to be $\mathcal{S}-$invariant.

Proposition 3.1. *Given a linear switching system* $\mathcal{S}$*, a subspace* $\bigcup_{i\in J} \{q_i\} \times \Omega_i$ *is* $\mathcal{S}-$*invariant if and only if for any* $i \in J$ *the following conditions hold:*

- $A_i\Omega_i \subset \Omega_i$*;*
- $R(e)\Omega_i \subset \Omega_h$*, for any* $e = (q_i, \sigma, q_h) \in E$*.*

Since the intersection of any two $\mathcal{S}-$invariant subspaces is an $\mathcal{S}-$invariant subspace, the minimal $\mathcal{S}-$invariant subspace containing a given subspace is well defined.

Let

$$\mathcal{G} = \bigcup_{i\in J} \{q_i\} \times \mathcal{G}_i \tag{6}$$

be the minimal $\mathcal{S}-$invariant subspace containing

$$\mathcal{H} = \bigcup_{i\in J} \{q_i\} \times \mathrm{Im}\,(B_i)\,.$$

For any $i \in J$, let

$$\mathcal{C}_i = \begin{pmatrix} B_i & A_iB_i & \ldots & A_i^{n_i-1}B_i \end{pmatrix}$$

be the controllability matrix associated with the linear system $S(q_i)$ and set

$$\mathcal{R} = \bigcup_{i\in J} \{q_i\} \times \mathcal{R}_i,$$

where $\mathcal{R}_i = \mathrm{Im}\,(\mathcal{C}_i)$. The following result holds.

Lemma 3.2. *The set* $\mathcal{G}$ *is the minimal* $\mathcal{S}-$*invariant subspace that contains the hybrid subspace* $\mathcal{R}$*.*

The following result illustrates a procedure for computing $\mathcal{G}$ in a finite number of steps.

Theorem 3.3. *Given $\mathcal{S}$, define the sequence of subspaces $\Omega_i^k \subset \mathbb{R}^{n_i}$, $k = 0, 1, 2, \ldots$, $i \in J$, as*

$$\begin{aligned} \Omega_i^0 &= \mathcal{R}_i, \\ \Omega_i^k &= \textstyle\sum_{h=0}^{n_i-1} (A_i)^h \, \Phi_i^k \\ \Phi_i^k &= \textstyle\sum_{j \in J_i} R((q_j, \sigma, q_i)) \Omega_j^{k-1} + \Omega_i^{k-1} \end{aligned}$$

where $J_i = \{j \in J : (q_j, \sigma, q_i) \in E\}$. The sequence $\{\Omega_i^k, i \in J\}_{k=0,1,2,\ldots}$ converges in $k^* \leq \sum_{i=1}^{N} n_i$ steps and

$$\mathcal{G} = \textstyle\bigcup_{i \in J} \{q_i\} \times \Omega_i^{k^*}.$$

By definition, the discrete evolution of the linear switching system $\mathcal{S} = (\Xi, \Theta, S, E, R)$ is described by the FSM (Q, Σ, E). We recall that the FSM is said to be *strongly connected* if there exists a path between any pair of discrete states in Q. By combining Proposition 3.1 and Theorem 3.3, the following result is obtained:

Proposition 3.4. *If $n_i = n$ for any $i \in J$, if $R(e) = I$ for any $e \in E$, and if (Q, Σ, E) is strongly connected, then*

$$\mathcal{G} = Q \times \widehat{\mathcal{G}},$$

where $\widehat{\mathcal{G}} \subset \mathbb{R}^n$ is the minimal linear subspace of $\mathbb{R}^n$ satisfying for any $i \in J$ the following conditions:

$$A_i \widehat{\mathcal{G}} \subset \widehat{\mathcal{G}}; \qquad \mathrm{Im}\,(B_i) \subset \widehat{\mathcal{G}}.$$

Remark 2. *The subspace $\widehat{\mathcal{G}}$ coincides with the 'multiple controllable subspace', as defined in [SUN 05] in the framework of switched linear systems.*

4. State space reductions based on stabilizability

It is well–known that a linear system S is asymptotically stabilizable if and only if a suitable subsystem extracted from S is asymptotically stable. In the context of general linear switching systems, stabilizability conditions become a bit more involved.

In this section, we show how to extract from a given linear switching system $\mathcal{S}$, a number of subsystems so that the stabilizability of some of them and the asymptotic stability of the remaining ones imply the stabilizability of $\mathcal{S}$. This reduces the computational effort required for checking stabilizability.

Our procedure is based on the reduction of the state space of the linear switching system $\mathcal{S}$ by means of the invariant hybrid subspace $\mathcal{G}$, as defined in the previous section.

Given the hybrid invariant subspace $\mathcal{G}$ as in (6), let $\mu_i \leq n_i$ be the dimension of $\mathcal{G}_i$ and define a hybrid state space transformation for $\mathcal{S}$ as follows. For each $i \in J$, consider the matrix:

$$T_i = \begin{pmatrix} b_1^i \dots b_{\mu_i}^i & v_1^i \dots v_{n_i-\mu_i}^i \end{pmatrix} \in \mathbb{R}^{n_i \times n_i},$$

where the vectors $b_1^i, \dots, b_{\mu_i}^i$ are a basis for $\mathcal{G}_i$ and the vectors $v_1^i, \dots, v_{n_i-\mu_i}^i$ are such that T_i is full rank. Then the matrices:

$$\begin{aligned} &\widehat{A}_i = T_i^{-1} A_i T_i, \\ &\widehat{B}_i = T_i^{-1} B_i, i \in J \\ &\widehat{R}(e) = T_h^{-1} R(e) T_i, e = (q_i, \sigma, q_h), \end{aligned}$$

take the form:

$$\begin{aligned} &\widehat{A}_i = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} \\ 0 & A_i^{(22)} \end{pmatrix}, \qquad \widehat{B}_i = \begin{pmatrix} B_i^{(1)} \\ 0 \end{pmatrix}, \\ &\widehat{R}(e) = \begin{pmatrix} R_e^{(11)} & R_e^{(12)} \\ 0 & R_e^{(22)} \end{pmatrix}, \end{aligned} \tag{7}$$

where $A_i^{(11)} \in \mathbb{R}^{\mu_i \times \mu_i}$. The linear switching system obtained after the hybrid state space transformation is algebraically equivalent [POL 06] to the linear switching system $\mathcal{S}$. Note that, in general, the pair $(A_i^{(11)}, B_i^{(1)})$ is not controllable.

We can define the following autonomous linear switching system (uncontrollable subsystem of $\mathcal{S}$):

$$\mathcal{S}_{un} = (\Xi_{un}, \Theta, S_{un}, E_{un}, R_{un}),$$

where:

– $\Xi_{un} = \bigcup_{i \in J_{un}} \{q_i\} \times \mathbb{R}^{n_i - \mu_i}$, $J_{un} = \{i \in J : \mu_i < n_i\}$;

– for any $i \in J_{un}$, $S_{un}(q_i)$ is described by the equation:

$$\dot{z}(t) = A_i^{(22)} z(t);$$

– $E_{un} = \{e = (q_i, \sigma, q_j) \in E : i, j \in J_{un}\}$;

– for any $e \in E_{un}$, $R_{un}(e) = R_e^{(22)}$.

The following result gives a relationship between stabilizability properties of $\mathcal{S}$ and stability properties of $\mathcal{S}_{un}$.

Theorem 4.1. *If $\mu_i < n_i$, $\forall i \in J$, the system $\mathcal{S}$ is asymptotically stabilizable only if $\mathcal{S}_{un}$ is asymptotically stable.*

A stronger result can be assessed under the following assumption:

Assumption 2 For any $i \in J$, $0 < \mu_i \leq n_i$.

Note that if $\mu_i = 0$, then $B_i = 0$ and any continuous state in $\mathcal{G}_h$ is reset to the origin after any transition of the form $(q_h, \sigma, q_i) \in E$.

Under Assumption 2, we can define the linear switching system (controlled subsystem of $\mathcal{S}$):

$$\mathcal{S}_c = (\Xi_c, \Theta, S_c, E, R_c),$$

where:

– $\Xi_c = \bigcup_{i \in J} \{q_i\} \times \mathbb{R}^{\mu_i}$;
– for any $i \in J$, $S_c(q_i)$ is described by the equation:

$$\dot{z}(t) = A_i^{(11)} z(t) + B_i^{(1)} u(t);$$

– for any $e \in E$, $R_c(e) = R_e^{(11)}$.

On the basis of the above decomposition, we now show that the asymptotic stabilizability of $\mathcal{S}$ can be reduced to the asymptotic stabilizability of $\mathcal{S}_c$ and the asymptotic stability of $\mathcal{S}_{un}$.

Theorem 4.2. *If Assumption 2 holds, then $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_c$ is asymptotically stabilizable and $\mathcal{S}_{un}$ is asymptotically stable.*

The result above clearly links to the classical *Kalman decomposition* of linear systems. We now show that the Kalman decomposition–based stabilizability characterization of linear systems can be extended to switching systems. We first need to introduce a particular class of controls. A control strategy φ is said to be a *static hybrid linear state feedback,* if for any discrete state $i \in J$, there exists a matrix $K_i \in \mathbb{R}^{m \times n_i}$ such that:

$$\begin{aligned} &\varphi(\eta|_{[0,t]}) = K_i x(t,j), \\ &\eta(t) = (q_i, x(t,j)). \end{aligned}$$

A switching system $\mathcal{S}$ is said to be *asymptotically stabilizable via static hybrid linear state feedback* if it is asymptotically stabilizable and the stabilizing control strategy is a static hybrid linear state feedback.

The following result shows that, under appropriate assumptions, the switching system $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_{un}$ is asymptotically stable.

Proposition 4.3. *If Assumption 2 holds and if*

$$\mathcal{G} = \mathcal{R}, \tag{8}$$

then $\mathcal{S}$ is asymptotically stabilizable if and only if $\mathcal{S}_{un}$ is asymptotically stable. Moreover, in this case, $\mathcal{S}$ is asymptotically stabilizable via static hybrid linear state feedback.

Even if condition (8) is not satisfied, some conditions on the switching system are given in [De 04b], under which asymptotic stabilizability of $\mathcal{S}$ is implied by asymptotic stability of $\mathcal{S}_{un}$.

We illustrate our result by means of a procedure that reduces step by step the computational effort required for checking stabilizability of linear switching systems. Here we assume that Assumption 2 holds for all the hybrid subspaces computed in the procedure.

In the following, controllable location means a discrete state $q_i \in Q$ whose associated linear system $S(q_i)$ is controllable. A strongly connected component of the linear switching system $\mathcal{S}$ is a linear switching subsystem whose FSM is a strongly connected component of the FSM associated with $\mathcal{S}$; such a system will be called maximal when its discrete state space is the maximal subset of Q having the property above.

Given a linear switching system $\mathcal{S} = (\Xi, \Theta, S, E, R)$, define the restriction of $\mathcal{S}$ to a subset Q' of Q as a linear switching system:

$$\mathcal{S}|_{Q'} = (\Xi', \Theta, S', E', R'), \tag{9}$$

where:

$$\begin{array}{l} \Xi' = \bigcup_{q_i \in Q'} \{q_i\} \times \mathbb{R}^{n_i}; \\ S'(q) = S(q), \forall q \in Q'; \\ E' = \{(q_i, \sigma, q_h) \in E : q_i, q_h \in Q'\}; \\ R'(e) = R(e), \forall e \in E'. \end{array}$$

Removing locations in $Q'' \subset Q$ from $\mathcal{S}$ means defining the restriction of $\mathcal{S}$ to $Q' = Q \backslash Q''$.

Procedure (Stabilizability-based Reduction)

1) Given a linear switching system $\mathcal{S}$, let Q_1 be the set of discrete states $q \in Q$ such that $S(q)$ is not controllable.

2) If $Q_1 = \varnothing$ then **STOP**: $\mathcal{S}$ **is controllable**. Otherwise let $\mathcal{S}_1$ be the restriction of $\mathcal{S}$ to Q_1.

3) Compute the maximal strongly connected components $\mathcal{F}_i$, $i \in J^1$, of $\mathcal{S}_1$ ($\mathcal{S}_1$ is asymptotically stabilizable if and only if each $\mathcal{F}_i$ is asymptotically stabilizable [De 06a]); let $J_{\mathcal{F}_i}$ be the index set associated with the discrete states of $\mathcal{F}_i$, for any $i \in J^1$.

4) Compute the invariant subspace $\mathcal{G}^{(i)} = \bigcup_{h \in J_{\mathcal{F}_i}} \{q_h\} \times \mathcal{G}_h^{(i)}$, for each strongly connected component $\mathcal{F}_i$. Let $\mathcal{S}_c^{(i)}$ be the controlled subsystem of $\mathcal{F}_i'$, $i \in J^1$, where $\mathcal{F}_i'$ is obtained by removing the locations q_h with $\mathcal{G}_h^{(i)} = \{0\}$ from $\mathcal{F}_i$.

5) If $\mathcal{S}_c^{(i)}$ is not asymptotically stabilizable for some $i \in J^1$, then **STOP**: $\mathcal{S}$ **is not asymptotically stabilizable**.

6) Remove the locations q_h, $h \in J_{\mathcal{F}_i}$, for which $\mathcal{G}_h^{(i)} = \mathbb{R}^{n_h}$ and $\mathcal{S}_c^{(i)}$ is asymptotically stabilizable (for any execution with initial discrete state q_h the hybrid state remains in $\mathcal{G}^{(i)}$, for any control action. Since $\mathcal{S}_c^{(i)}$ is asymptotically stabilizable, then q_h can be removed [De 06a]). Let Q_2 be the reduced discrete state space.

7) If $Q_2 = \varnothing$ then **STOP**: $\mathcal{S}$ **is asymptotically stabilizable**. Otherwise let $\mathcal{S}_2$ be the restriction of $\mathcal{S}_1$ to Q_2.

8) Compute the maximal strongly connected components $\widetilde{\mathcal{F}}_i$, $i \in J^2$, of $\mathcal{S}_2$.

9) Compute the invariant subspace $\widetilde{\mathcal{G}}^{(i)}$, for each $\widetilde{\mathcal{F}}_i$. Let $\widetilde{\mathcal{S}}_{un}^{(i)}$, $i \in J^2$ be the uncontrolled subsystems of $\widetilde{\mathcal{F}}_i$.

10) **STOP**: Return $\{\mathcal{S}_c^{(i)}, i \in J^1\}$ and $\{\widetilde{\mathcal{S}}_{un}^{(i)}, i \in J^2\}$.

Since controllability implies stabilizability, and controllability is easy to check (cf. Remark 1), in the following theorem we assume that $\mathcal{S}$ is not controllable.

Theorem 4.4. *The stabilizability-based reduction procedure converges in a finite number of steps. A noncontrollable linear switching system $\mathcal{S}$ is asymptotically stabilizable if and only if the linear switching system $\mathcal{S}_c^{(i)}$ is asymptotically stabilizable $\forall i \in J^1$ and the linear switching system $\widetilde{\mathcal{S}}_{un}^{(i)}$ is asymptotically stable $\forall i \in J^2$.*

This last theorem decomposes the problem of checking stabilizability of a given linear switching system into simpler subproblems. In particular, the given system is decoupled into controlled and autonomous linear switching subsystems. The asymptotic stabilizability of the first ones and the asymptotic stability of the latter ones imply the asymptotic stabilizability of the given system.

5. Conclusions

In this paper, we considered affine switching systems and proposed some state space decompositions, based on hybrid invariant subspaces, which yield a complexity reduction in checking stabilizability. The given system is decoupled into controlled and autonomous linear switching subsystems. The asymptotic stabilizability of the first ones and the asymptotic stability of the latter ones imply (and are implied by) the asymptotic stabilizability of the given system.

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